

# XXX. 傅里叶变换初步

## 1. 傅里叶级数

我们从数学分析中学到了以下定理:

**定理 1.** 如果  $f(x)$  是区间  $(a, b)$  中的分段平滑函数, 并且对于  $(a, b)$  中的每个点  $x$ ,  $f(x) = (f(x^-) + f(x^+))/2$ , 其中  $f(x^-)$  和  $f(x^+)$  分别表示  $f(x)$  在  $x$  的左右极限, 那么对于  $(a, b)$  中的每个点  $x$ ,

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{b-a}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2\pi n x}{b-a} + b_n \sin \frac{2\pi n x}{b-a} \right) \end{aligned} \quad (1)$$

其中,

$$\begin{aligned} c_n &= \frac{1}{b-a} \int_a^b f(x) \exp\left(\frac{-2\pi i n x}{b-a}\right) dx \quad (n = 0, \pm 1, \pm 2, \dots) \\ a_n &= \frac{2}{b-a} \int_a^b f(x) \cos\left(\frac{2\pi n x}{b-a}\right) dx \quad (n = 0, 1, 2, \dots) \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin\left(\frac{2\pi n x}{b-a}\right) dx \quad (n = 1, 2, \dots) \end{aligned}$$

## 2. 连续的傅里叶变换

假设函数在  $(-\infty, \infty)$  中是绝对可积的, 在傅立叶展开式(1)中, 令  $a = -L$  和  $b = L$ , 则

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{i n \pi x}{L}\right) \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2L} \exp\left(\frac{i n \pi x}{L}\right) \int_{-L}^L f(\xi) e^{-i \frac{\pi n \xi}{L}} d\xi \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \frac{\pi}{L} \exp\left(i x \frac{n\pi}{L}\right) \int_{-L}^L f(\xi) e^{-i \frac{\pi n \xi}{L}} d\xi \\ &\xrightarrow{\Delta\omega=\pi/L, \omega_n=n\pi/L} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(ix\omega) \int_{-\infty}^{\infty} f(\xi) e^{-i\xi\omega} d\xi, \quad \text{当 } L \rightarrow \infty \end{aligned}$$

其中,

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt \quad (2)$$

被称为函数  $f(t)$  的傅里叶变换.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) \exp(i\omega t) d\omega \quad (3)$$

被称为函数  $f(t)$  的逆傅里叶变换.

### 3. 离散的傅里叶变换

考虑一个截断函数  $f(t)$ , 它从  $t = 0$  到  $T_f > 0$  是有限的, 在其他地方是 0. 我们将  $(0, T_f)$  等分为  $N$  段. 记

$$f_k = f((k-1)\Delta), \quad \Delta = \frac{T_f}{N}, \quad k = 1, \dots, N,$$

于是由傅里叶变换(2)和逆傅里叶变换(3)得

$$\begin{aligned} \tilde{f}\left(\omega = \frac{2\pi(n-1)}{N\Delta}\right) &= \int_{-\infty}^{\infty} f(x) \exp\left(-i\frac{2\pi(n-1)}{N\Delta}x\right) dx \\ &\approx \Delta \sum_{k=1}^N f((k-1)\Delta) \exp\left(-i\frac{2\pi(n-1)(k-1)}{N}\right) \end{aligned} \quad (4)$$

$$\begin{aligned} f((k-1)\Delta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) \exp(i\omega(k-1)\Delta) d\omega \\ &\approx \frac{1}{2\pi} \sum_{n=1}^N \tilde{f}\left(\frac{2\pi(n-1)}{N\Delta}\right) \frac{2\pi}{N\Delta} \exp\left(i\frac{2\pi(n-1)(k-1)}{N}\right) \\ &= \frac{1}{N} \sum_{n=1}^N \tilde{f}\left(\frac{2\pi(n-1)}{N\Delta}\right) \frac{1}{\Delta} \exp\left(i\frac{2\pi(n-1)(k-1)}{N}\right) \end{aligned} \quad (5)$$

记

$$\tilde{f}_n = \tilde{f}\left(\frac{2\pi(n-1)}{N\Delta}\right) / \Delta,$$

则我们可以把(4)和(5)写成离散的傅里叶变换对:

$$\tilde{f}_n = \sum_{k=1}^N f_k \exp\left(-i\frac{2\pi(n-1)(k-1)}{N}\right) \quad (6)$$

$$f_k = \frac{1}{N} \sum_{n=1}^N \tilde{f}_n \exp\left(i\frac{2\pi(n-1)(k-1)}{N}\right) \quad (7)$$

或者写成矩阵形式:

$$\begin{aligned}
& \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{(-2\pi i * 1 * 1 / N)} & e^{(-2\pi i * 1 * 2 / N)} & \cdots & e^{(-2\pi i * 1 * (N-1) / N)} \\ 1 & e^{(-2\pi i * 2 * 1 / N)} & e^{(-2\pi i * 2 * 2 / N)} & \cdots & e^{(-2\pi i * 2 * (N-1) / N)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{(-2\pi i * (N-1) * 1 / N)} & e^{(-2\pi i * (N-1) * 2 / N)} & \cdots & e^{(-2\pi i * (N-1) * (N-1) / N)} \end{bmatrix} \\
& \times \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{bmatrix} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ \vdots \\ \tilde{f}_N \end{bmatrix} \\
& \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{(2\pi i * 1 * 1 / N)} & e^{(2\pi i * 1 * 2 / N)} & \cdots & e^{(2\pi i * 1 * (N-1) / N)} \\ 1 & e^{(2\pi i * 2 * 1 / N)} & e^{(2\pi i * 2 * 2 / N)} & \cdots & e^{(2\pi i * 2 * (N-1) / N)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{(2\pi i * (N-1) * 1 / N)} & e^{(2\pi i * (N-1) * 2 / N)} & \cdots & e^{(2\pi i * (N-1) * (N-1) / N)} \end{bmatrix} \\
& \times \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{bmatrix}
\end{aligned}$$

这就是快速傅里叶算法.

## 4. 傅里叶变换的应用

### 4.1 $\delta$ -函数的性质

**定义.**  $\delta$ -函数也称为狄拉克函数或脉冲函数, 它被定义为

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad (8)$$

并且  $\int \delta(x) dx = 1$ .

#### 性质

$$(1) \quad \delta(x) = \frac{1}{2\pi} \int dk e^{ikx}, i^2 = -1.$$

$$(2) \quad \text{对于任何连续函数 } f(x), \int_{-\infty}^{+\infty} f(x) \delta(x-y) dx = f(y).$$

证明.

(1).

记  $I_\delta(x) \equiv \frac{1}{2\pi} \int dk e^{ikx}$ . 对于  $x = 0$ ,  $I_\delta(x) = \infty$ . 对于  $x \neq 0$ ,

$$I_\delta(x) = \frac{1}{2\pi} \int dk \cos(kx) + \frac{i}{2\pi} \int dk \sin(kx)$$

由于两个积分函数都是振荡函数,  $I_\delta(x) = 0$ . 并且,

$$\begin{aligned} \int_{-A}^A dx I_\delta(x) &= \frac{1}{2\pi} \int dk \int_{-A}^A e^{ikx} dx \\ &= \frac{1}{\pi} \int \frac{\sin kA}{kA} dk A \\ &= \frac{2}{\pi} \int_0^\infty dt \frac{\sin t}{t} = 1 \quad \left( \because \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2} \right) \end{aligned}$$

所以  $I_\delta(x) = \delta(x)$ .

(2).

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int e^{-ikx} \tilde{f}(k) dk \\ &= \int dk e^{-ikx} \int dy e^{iky} f(y) \\ &= \int dy f(y) \left[ \frac{1}{2\pi} \int dk e^{-ik(x-y)} \right] \\ &= \int dy f(y) \delta(x-y) \end{aligned}$$

## 4.2. 自相关函数的计算

考虑一个截断函数  $f(t)$ , 它从  $t = 0$  到  $T_f > 0$  是有限的, 在其他地方是 0.

其自相关函数定义为

$$g(\tau) = \frac{1}{T_f} \int_{-\infty}^{+\infty} f^*(t) f(t + \tau) dt$$

为了很好地定义  $g(\tau)$ , 应该选择  $\tau \ll T_f$ . 相关长度定义为  $\tau_{\max} \ll T_f$ , 高于该长度时,  $g(\tau)$  近似为零.

因为

$$\begin{aligned}
g(\tau) &= \frac{1}{T_f} \int_{-\infty}^{+\infty} f^*(t) f(t + \tau) dt \\
&= \frac{1}{T_f} \int_{-\infty}^{+\infty} dt \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\omega)^* e^{i\omega t} d\omega \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega') e^{-i\omega'(t+\tau)} d\omega' \right] \\
&= \frac{1}{4\pi^2 T_f} \int_{-\infty}^{+\infty} \tilde{f}(\omega)^* d\omega \int_{-\infty}^{+\infty} F(\omega') e^{-i\omega'\tau} d\omega' \int_{-\infty}^{+\infty} e^{i(\omega-\omega')t} dt \\
&= \frac{1}{2\pi T_f} \int_{-\infty}^{+\infty} \tilde{f}(\omega)^* d\omega \int_{-\infty}^{+\infty} F(\omega') e^{-i\omega'\tau} d\omega' \delta(\omega - \omega') \\
&= \frac{1}{2\pi T_f} \int_{-\infty}^{+\infty} \tilde{f}(\omega)^* \tilde{f}(\omega) e^{-i\omega\tau} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\tilde{f}(\omega)|^2}{T_f} e^{-i\omega\tau} d\omega
\end{aligned}$$

所以

$$\tilde{g}(\omega) = \frac{|\tilde{f}(\omega)|^2}{T_f} \quad (9)$$

这意味着自相关函数  $g(\tau)$  的傅里叶谱  $\tilde{g}(\omega)$  就是原信号  $f(t)$  谱的功率密度： $|\tilde{f}(\omega)|^2/T_f$ .

(9) 被称为简化的 Wiener-Khinchin 定理.

考虑自相关函数的离散形式

$$g_j = \frac{1}{N} \sum_{k=1}^N f_k^* f_{k+j-1}, \quad j = 1, \dots, N$$

这里假设  $f$  以  $N$  为周期，即  $f_{k+j-1>N} = f_{k+j-1-N}$ .

它的傅里叶变换是

$$\tilde{g}_n = \sum_{j=1}^N g_j \exp\left(-i \frac{2\pi(n-1)(j-1)}{N}\right) = \frac{|\tilde{f}_n|^2}{N}. \quad (10)$$

我们先以  $N = 3$  为例来说明.

$$\begin{aligned}
g_1 &= \frac{1}{N}(f_1^* f_1 + f_2^* f_2 + f_3^* f_3), \quad j = 1 \\
g_2 &= \frac{1}{N}(f_1^* f_2 + f_2^* f_3 + f_3^* f_4) \\
&= \frac{1}{N}(f_1^* f_2 + f_2^* f_3 + f_3^* f_1), \quad j = 2 \\
g_3 &= \frac{1}{N}(f_1^* f_3 + f_2^* f_4 + f_3^* f_5) \\
&= \frac{1}{N}(f_1^* f_2 + f_2^* f_1 + f_3^* f_2), \quad j = 3
\end{aligned}$$

证明.

$$\begin{aligned}
\tilde{g}_n &= \sum_{j=1}^N g_j \exp\left(-i \frac{2\pi(n-1)(j-1)}{N}\right) \\
&= \sum_{j=1}^N \left( \frac{1}{N} \sum_{k=1}^N f_k^* f_{k+j-1} \right) \exp\left(i \frac{2\pi(n-1)(k-1)}{N}\right) \exp\left(-i \frac{2\pi(n-1)(k+j-2)}{N}\right) \\
&= \frac{1}{N} \sum_{k=1}^N f_k^* \exp\left(i \frac{2\pi(n-1)(k-1)}{N}\right) \sum_{j=1}^N f_{k+j-1} \exp\left(-i \frac{2\pi(n-1)(k+j-2)}{N}\right) \\
&= \frac{1}{N} \sum_{k=1}^N f_k^* \exp\left(i \frac{2\pi(n-1)(k-1)}{N}\right) \left( \sum_{j'=k}^{k+N-1} f_{j'} \exp\left(-i \frac{2\pi(n-1)(j'-1)}{N}\right) \right) \\
&= \frac{1}{N} \tilde{f}_n \sum_{k=1}^N f_k^* \exp\left(i \frac{2\pi(n-1)(k-1)}{N}\right) \quad (\because \text{periodicity}) \\
&= \frac{1}{N} |\tilde{f}_n|^2
\end{aligned}$$

于是我们在离散的情况下也证明了自相关函数的谱  $\tilde{g}_n$  是原信号谱的功率密度  $\frac{1}{N} |\tilde{f}_n|^2$ .

### 4.3 自相关函数的积分

令  $H(t)$  为单位阶跃函数或称为 Heaviside 函数:

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

则

$$\begin{aligned}
\int_0^\infty e^{-i\omega\tau} d\tau &= \lim_{\omega_0 \rightarrow 0^+} \int_0^\infty e^{-(\omega_0+i\omega)\tau} d\tau \\
&= \lim_{\omega_0 \rightarrow 0^+} \frac{1}{\omega_0 + i\omega} \\
&= \lim_{\omega_0 \rightarrow 0^+} \left( \frac{\omega_0}{\omega_0^2 + \omega^2} - \frac{i\omega}{\omega_0^2 + \omega^2} \right) \\
&= \pi\delta(\omega) - i\mathcal{P} \frac{1}{\omega}
\end{aligned}$$

所以,

$$\begin{aligned}
\int_0^\infty g(\tau) d\tau &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\tilde{f}(\omega)|^2}{T_f} d\omega \int_0^\infty e^{-i\omega\tau} d\tau \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\tilde{f}(\omega)|^2}{T_f} \left( \pi\delta(\omega) - i\mathcal{P} \frac{1}{\omega} \right) d\omega \\
&= \lim_{\omega_0 \rightarrow 0} \frac{|\tilde{f}(\omega_0)|^2}{2T_f} \quad (\because g(\tau) \text{ 是实函数}, \omega \neq 0)
\end{aligned}$$

#### 4.4 示例

示例 1 (连续的情况). 考虑随机相位余弦曲线:

$$v(t) = A\cos(\omega_0 t + \phi)$$

其中  $\phi$  是一个均匀分布在  $(0, 2\pi)$  中的随机变量.

$v(t)$  的自相关函数为(整体平均)

$$\begin{aligned}
R_v(t_1, t_2) &= \mathbb{E}[\cos(\omega_0 t_1 + \phi)\cos(\omega_0 t_2 + \phi)] \\
&= \frac{A^2}{2\pi} \int_0^{2\pi} \cos(\omega_0 t_1 + \phi)\cos(\omega_0 t_2 + \phi) d\phi \quad (\because \phi \text{ 在 } (0, 2\pi) \text{ 中均匀分布}) \\
&= \frac{A^2}{4\pi} \int_0^{2\pi} [\cos(\omega_0(t_1 + t_2) + 2\phi) + \cos(\omega_0(t_2 - t_1))] d\phi \\
&= \frac{A^2}{2} \cos(\omega_0(t_2 - t_1)) = \frac{A^2}{2} \cos(\omega_0\tau), \quad \tau = t_2 - t_1
\end{aligned}$$

自相关函数的傅里叶变换为

$$\begin{aligned}
\frac{A^2}{2} \int \cos(\omega_0 t) e^{it\omega} dt &= \frac{A^2}{4} \int [e^{i(\omega+\omega_0)t} + e^{i(\omega-\omega_0)t}] dt \\
&= \frac{\pi A^2}{2} [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]
\end{aligned} \tag{11}$$

$v(t)$  傅里叶变换是

$$\begin{aligned}
\tilde{v}(\omega) &= \frac{A}{2} \int_{-\infty}^{\infty} (e^{i(\omega_0 t + \phi)} + e^{-i(\omega_0 t + \phi)}) e^{-i\omega t} dt \\
&= \frac{A}{2} e^{i\phi} \int_{-\infty}^{\infty} e^{i(\omega_0 - \omega)t} dt + \frac{A}{2} e^{-i\phi} \int_{-\infty}^{\infty} e^{-i(\omega_0 + \omega)t} dt \\
&= \pi A e^{i\phi} \delta(\omega_0 - \omega) + \pi A e^{-i\phi} \delta(\omega_0 + \omega)
\end{aligned}$$

$v(t)$  的谱功率密度为

$$\begin{aligned}
\frac{|\tilde{v}(\omega)|^2}{2\pi} &= \frac{\pi A^2}{2} [e^{-i\phi} \delta(\omega_0 - \omega) + e^{i\phi} \delta(\omega_0 + \omega)] [e^{i\phi} \delta(\omega_0 - \omega) + e^{-i\phi} \delta(\omega_0 + \omega)] \\
&= \frac{\pi A^2}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \quad (\because \delta(x)\delta(x) = \delta(x))
\end{aligned} \tag{12}$$

所以自相关函数的谱(11)等于原信号的谱功率密度(12). 这正是Wiener-Khinchin定理.

示例 2 (离散的情况). 下面, 我们用数值方法来检验Wiener-Khinchin定理.

```

# generate data
set.seed(12345)
A <- 5
w0 <- 2 * pi
Tf <- 10 * pi
N <- 1000
dt <- Tf/N
tk <- (0:(N-1)) * dt
phi <- runif(N, 0.0, 1.0) * 2 * pi
fk <- A * cos(w0 * tk + phi)
# plot data
plot(tk, fk, type='l', xlab='Time:s', ylab='Signal')

```

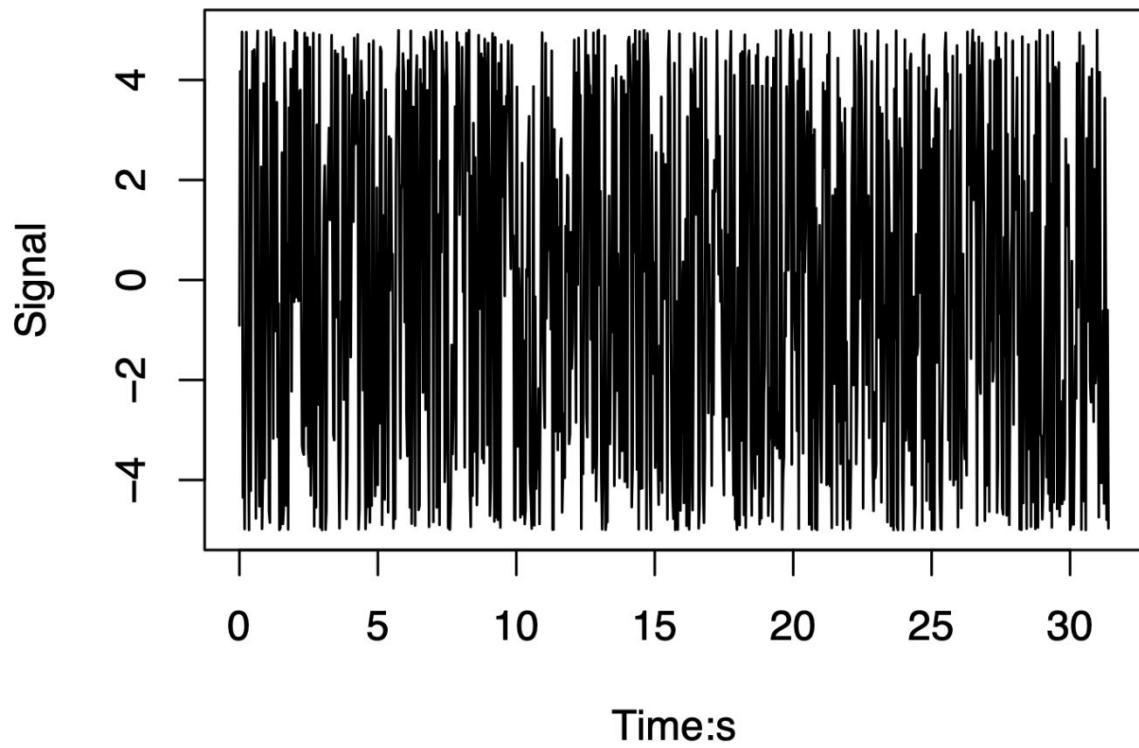


Figure 1. The raw signal.

```
# compute power spectral density
tildeFk <- fft(fk)
psd <- Re(Conj(tildeFk) * tildeFk)/N
df <- (0:(N-1))*2*pi/N/dt
plot(df, psd, type="l", xlab = 'Frequency')
```

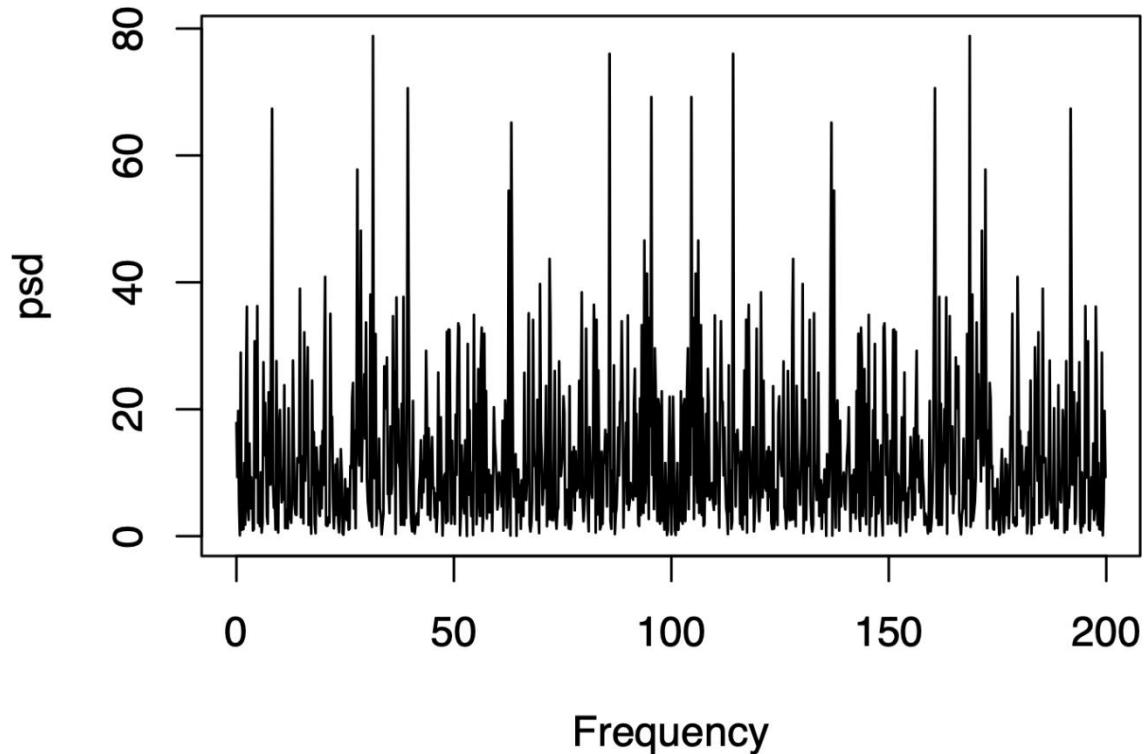


Figure 2. The power spectral density.

```
# compare power spectral density with spectrum of correlation function
g <- convolve(fk, fk, type = "circular")/length(fk)
tildeg <- Re(fft(g))
par(mar=c(5, 4, 2, 4))
par(xpd=FALSE)
plot(df, psd, type="l", xlab = 'Frequency')
points(df, tildeg, col='red')
par(xpd=TRUE)
legend(207, 80, legend = c("psd",expression(tilde(g))),pch=c(NA, 1), lty =
c(1, NA), cex = 0.9, bty = 'n', col=c(1,2))
```

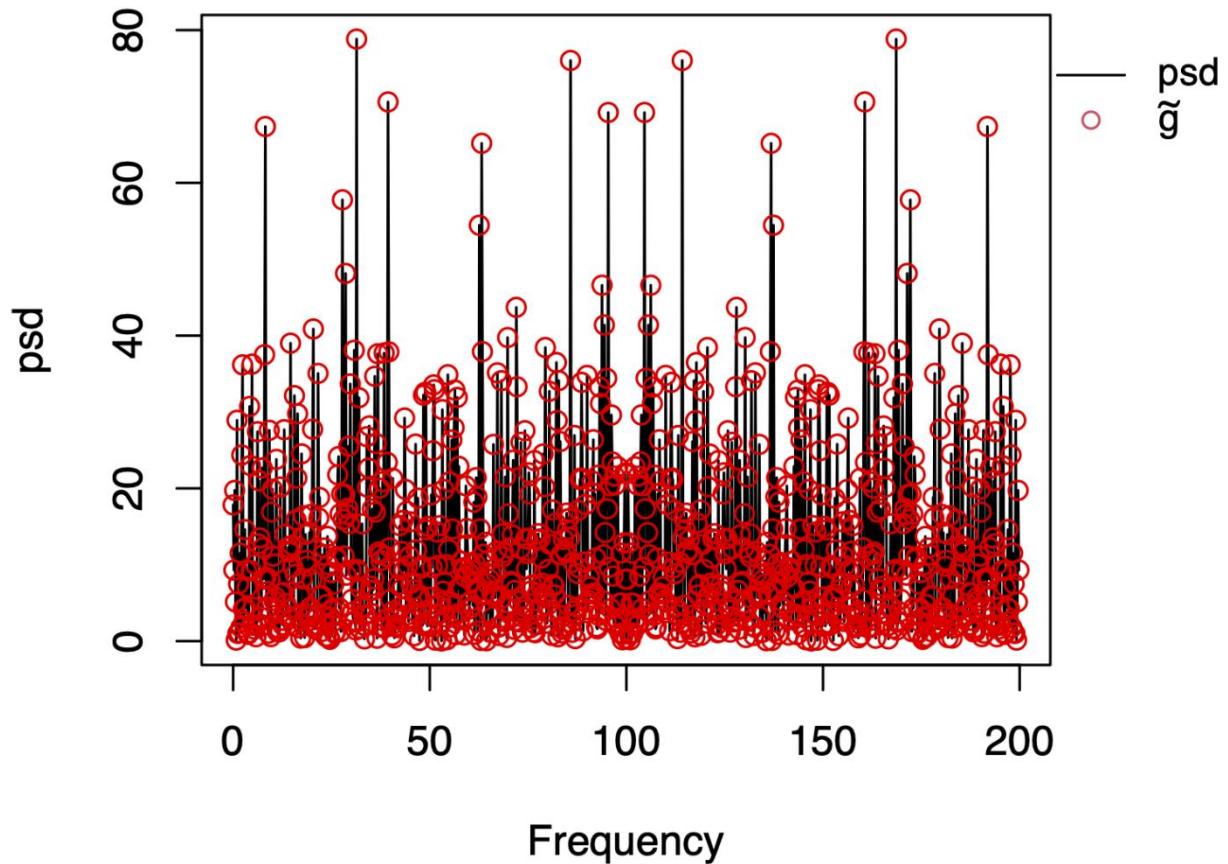


Figure 3. Comparison between PSD and the spectrum of correlation function.

图3清楚地表明Wiener-Khinchin定理成立。

上面的自相关函数的计算用到信号周期重复的假设。

如果我们用有限信号数据来定义自相关函数，情况会怎样呢？

```
# correlation function defined by finite data
corFunc <- function(Jx) {
  n_Jx <- length(Jx)
  n_Jx1 <- n_Jx - 1
  c_Jx <- c(mean(Jx^2))
  if (n_Jx1 > 0) {
    for (i in 1:n_Jx1) {
      c_Jx[i+1] <- mean(Jx[1:(n_Jx - i)] * Jx[-(1:i)])
    }
  }
  return(c_Jx)
}
```

```

# compare psd with spectrum of correlation function from finite data
g <- corFunc(fk)
tildeg <- Re(fft(g))
par(mar=c(5, 4, 2, 4))
par(xpd=FALSE)
plot(df, psd, type="l", xlab = 'Frequency')
points(df, tildeg, col='red')
par(xpd=TRUE)
legend(207, 80, legend = c("psd",expression(tilde(g))), pch=c(NA, 1), lty =
c(1, NA), cex = 0.9, bty = 'n', col=c(1,2))

```

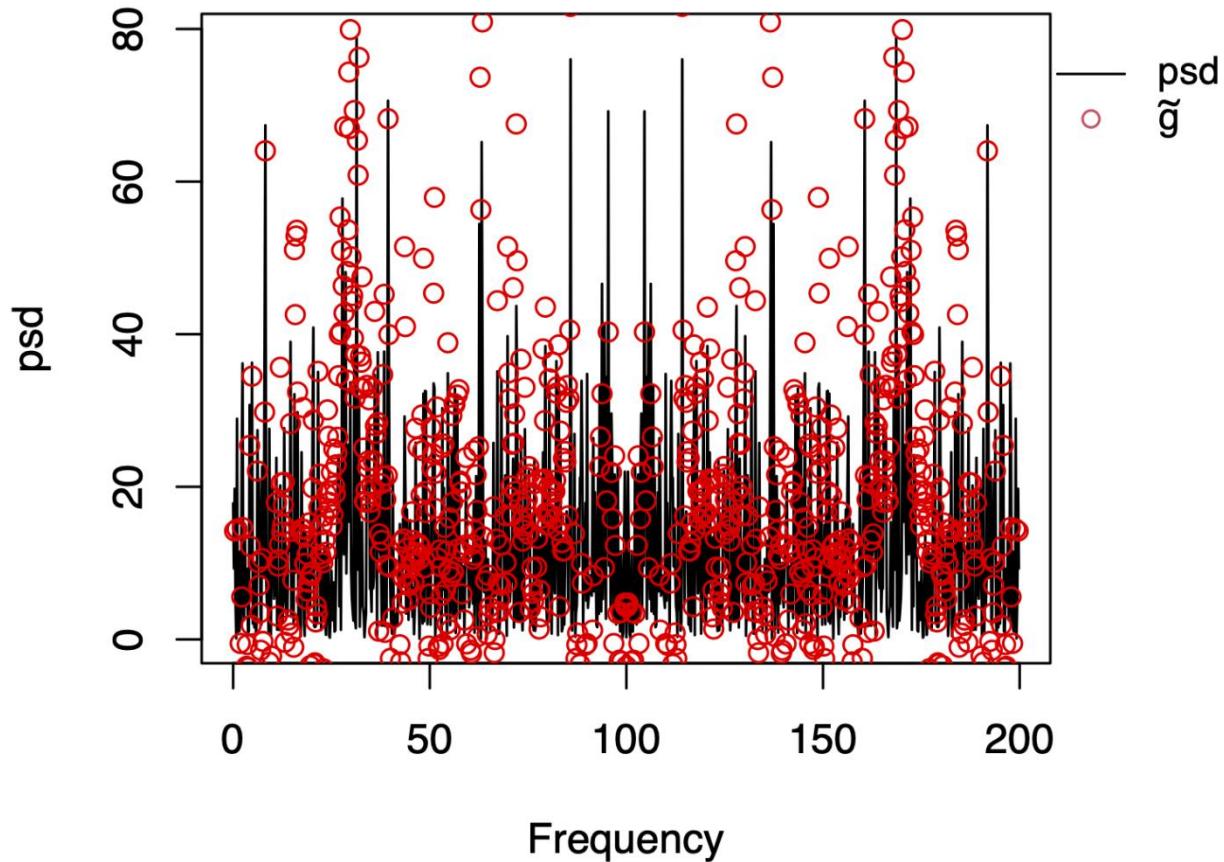


Figure 4. Comparison of PSD with the spectrum of correlation function calculated from finite data.

图4清楚地表明对于用有限信号数据定义的自相关函数, Wiener-Khinchin定理不成立.

示例 3 (相关函数的比较). 考虑离散数据系列:

$$a_0 = 0, \quad a_i = a_{i-1}e^{-1/L} + R_i, \quad i = 1, 2, 3, \dots$$

其中  $R_i$  是均匀分布在  $(-0.5, 0.5)$  上的独立随机数,  $L$  是表示特征相关长度的任意常数. 则数据系列  $\{a_i\}$  的相关函数为

$$g_k = \langle a_i a_{i+k-1} \rangle = \frac{e^{-(k-1)/L}}{12(1 - e^{-2/L})}$$

证明.

因为  $R_i$  是均匀分布在  $(-0.5, 0.5)$  上的独立随机数, 所以  $\langle a_{i-1}R_i \rangle = 0$ . 于是,

$$\begin{aligned} \langle a_i^2 \rangle &= \left\langle (a_{i-1}e^{-1/L} + R_i)^2 \right\rangle \\ &= \langle R_i^2 \rangle + \langle R_{i-1}^2 \rangle e^{-2/L} + \langle R_{i-2}^2 \rangle e^{-4/L} + \dots \\ &= \frac{\langle R^2 \rangle}{1 - e^{-2/L}} \\ &= \frac{1}{1 - e^{-2/L}} \int_{-0.5}^{0.5} x^2 dx = \frac{1}{12(1 - e^{-2/L})} \end{aligned}$$

和

$$\begin{aligned} g_k &= \langle a_i a_{i+k-1} \rangle \\ &= \langle a_i (a_{i+k-2}e^{-1/L} + R_{i+k-1}) \rangle \\ &= \langle a_i a_{i+k-2} \rangle e^{-1/L} \\ &= \dots \\ &= \langle a_i a_i \rangle e^{-(k-1)/L} = \frac{e^{-(k-1)/L}}{12(1 - e^{-2/L})} \end{aligned}$$

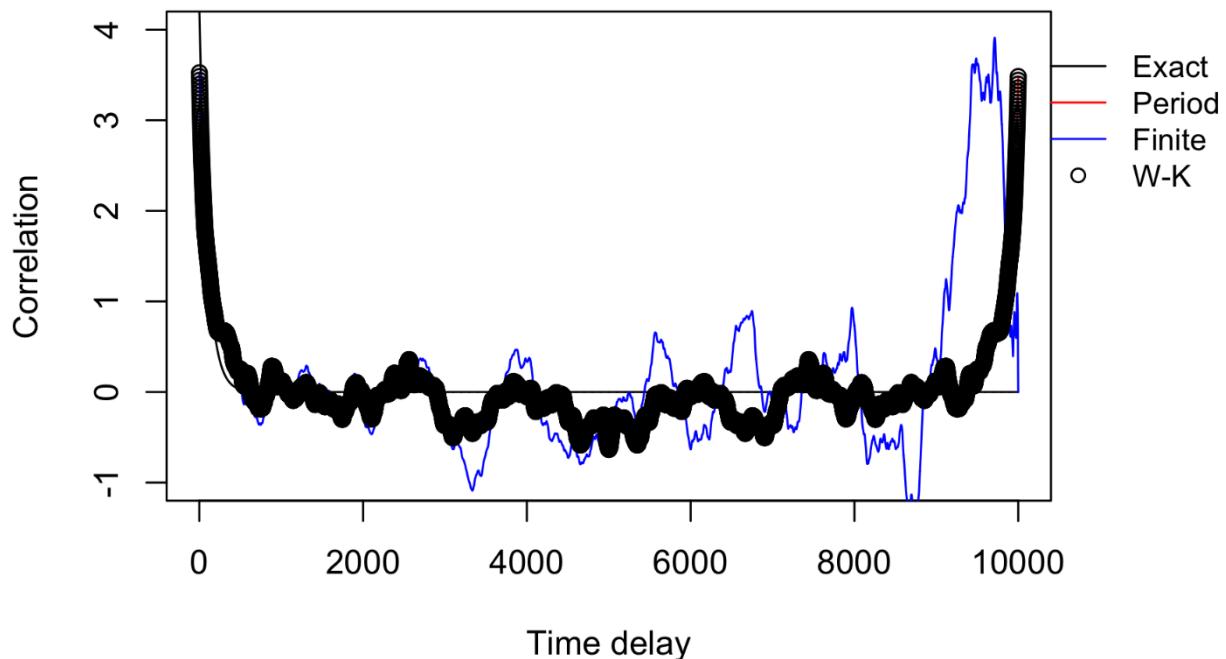
考虑  $N = 10^4$  和  $N/L = 100$  的情况

```
# generate data
set.seed(12345)
N <- 10^4
invL <- -100.0/N
x <- exp(invL)
a <- c(0)
for (i in 2:N) {
  a[i] <- a[i-1]*x + runif(1, -0.5, 0.5)
}
```

```

# analytic correlation function
analyticCor <- lapply(1:N, function(k) {x^(k-1)/12/(1-x^2)})
# periodic correlation function
g <- convolve(a, a, type = "circular")/length(a)
# finite size correlation function
z <- corFunc(a)
# Wiener-Khinchin theorem
tildeFk <- fft(a)
psd <- Re(Conj(tildeFk) * tildeFk)/N
sdCor <- Re(fft(psd, inverse = TRUE))/N
par(mar=c(5, 4, 2, 5))
par(xpd=FALSE)
plot(1:N, analyticCor, type="l", xlab = 'Time delay', ylab = 'Correlation',
ylim = c(-1,4))
lines(1:N, g, col="red")
lines(1:N, z, col="blue")
points(1:N, sdCor)
par(xpd=TRUE)
legend(10400, 4, legend = c("Exact", "Period", "Finite", "W-K"), lty =
c(1,1,1,NA), pch=c(NA,NA,NA,1), cex = 0.9, bty = 'n',
col=c("black", "red", "blue", "black")))

```



*Figure 5. Comparison of correlation functions calculated using different algorithms.*

上面的比较表明在计算自相关函数时，R中的算法convolve(a, a, type = "circular")/length(a)得到一致的结果。

## 5. 拉普拉斯变换

傅里叶变换有其局限性，例如不适用于解齐次方程

$$\frac{d^2y(t)}{dt^2} - y(t) = 0$$

因为该方程的通解是 $y(t) = c_1 e^t + c_2 e^{-t}$ ，但 $e^t$ 和 $e^{-t}$ 的傅里叶变换不存在。在这种情况下，我们可以将傅里叶变换应用于修改后的函数

$$Y(t) = y(t)e^{-\gamma t}H(t) \quad (13)$$

这里 $H(t)$ 是Heaviside阶梯函数，即当 $t \geq 0$ ， $H(t) = 1$ ；否则， $H(t) = 0$ 。 $\gamma$ 是任意足够大的实数以保证 $Y(t)$ 的傅里叶变换存在。

函数 $Y(t)$ 的傅里叶变换为：

$$\hat{Y}(\xi) = \int_{-\infty}^{\infty} Y(t)e^{-i\xi t} dt = \int_0^{\infty} y(t)e^{-\gamma t-i\xi t} dt = \int_0^{\infty} y(t)e^{-st} dt, \quad s = \gamma + i\xi \quad (14)$$

若将上述等式的右边项定义为函数 $y(t)$ 的拉普拉斯变换：

$$\tilde{y}(s) := \int_0^{\infty} y(t)e^{-st} dt \quad (15)$$

则 $\hat{Y}(\xi) = \tilde{y}(\gamma + i\xi)$ ，函数 $\hat{Y}(\xi)$ 的逆傅里叶变换为：

$$Y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{Y}(\xi) e^{i\xi t} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{y}(\gamma + i\xi) e^{i\xi t} d\xi \quad (16)$$

当 $t \geq 0$ ，变成

$$y(t)e^{-\gamma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{y}(\gamma + i\xi) e^{i\xi t} d\xi \quad (17)$$

即，

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{y}(\gamma + i\xi) e^{(\gamma+i\xi)t} d\xi = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{y}(s) e^{st} ds \quad (18)$$

上式为函数 $y(t)$ 的逆拉普拉斯变换。

## 参考文献

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