从直角坐标到球坐标的拉普拉斯算子

在直角坐标中，拉普拉斯算子定义为：

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

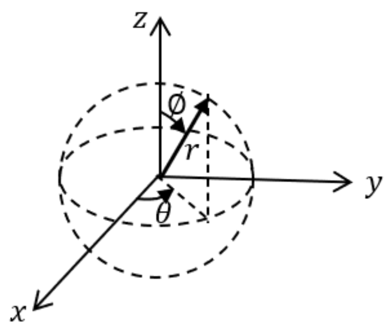

图 1: 球坐标示意图.

如图 1 所示，引入球坐标:

$$\begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi \end{aligned}$$

因为

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arctan(y/x) \\ \phi &= \arccos(z/r) \end{aligned}$$

所以

$$\frac{\partial r}{\partial x} = x/r = \sin\phi\cos\theta$$
$$\frac{\partial r}{\partial y} = y/r = \sin\phi\sin\theta$$
$$\frac{\partial r}{\partial z} = z/r = \cos\phi$$
$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r\sin\phi}$$
$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r\sin\phi}$$
$$\frac{\partial \theta}{\partial z} = 0$$
$$\frac{\partial \phi}{\partial x} = \frac{1}{r}\cos\phi\cos\theta$$
$$\frac{\partial \phi}{\partial y} = \frac{1}{r}\cos\phi\sin\theta$$
$$\frac{\partial \phi}{\partial z} = -\frac{\sin\phi}{r}$$

由此,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} \\ &= \sin\phi\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r\sin\phi}\frac{\partial f}{\partial \theta} + \frac{1}{r}\cos\phi\cos\theta\frac{\partial f}{\partial \phi}\end{aligned}$$

$$\frac{\partial^2 f}{\partial x^2} = \left[\sin\phi\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r\sin\phi}\frac{\partial}{\partial \theta} + \frac{1}{r}\cos\phi\cos\theta\frac{\partial}{\partial \phi}\right]\left[\sin\phi\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r\sin\phi}\frac{\partial f}{\partial \theta} + \frac{1}{r}\cos\phi\cos\theta\frac{\partial f}{\partial \phi}\right] \qquad (1)$$

类似地,

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} \\ &= \sin\phi\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r\sin\phi}\frac{\partial f}{\partial \theta} + \frac{1}{r}\cos\phi\sin\theta\frac{\partial f}{\partial \phi}\end{aligned}$$

$$\frac{\partial^2 f}{\partial y^2} = \left[\sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r\sin\phi}\frac{\partial}{\partial \theta} + \frac{1}{r}\cos\phi\sin\theta\frac{\partial}{\partial \phi}\right]\left[\sin\phi\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r\sin\phi}\frac{\partial f}{\partial \theta} + \frac{1}{r}\cos\phi\sin\theta\frac{\partial f}{\partial \phi}\right] \qquad (2)$$

$$
\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z} \\
&= \cos\phi \frac{\partial f}{\partial r} - \frac{1}{r}\sin\phi \frac{\partial f}{\partial \phi}
\end{aligned}
$$

$$
\frac{\partial^2 f}{\partial z^2} = \left[\cos\phi \frac{\partial}{\partial r} - \frac{1}{r}\sin\phi \frac{\partial}{\partial \phi}\right]\left[\cos\phi \frac{\partial f}{\partial r} - \frac{1}{r}\sin\phi \frac{\partial f}{\partial \phi}\right] \tag{3}
$$

将（1）、（2） 和 （3） 相加并简化, 可以得到球坐标中的拉普拉斯算子:

$$
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial f}{\partial r}\right] + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left[\sin\phi \frac{\partial f}{\partial \phi}\right] + \frac{1}{r^2 \sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2} \tag{4}
$$