Martingales

Probability space

Definition 1. A probability space is defined by three elements Ω, $\mathcal{F}$, and $\mathcal{P}$. Ω is the sample space, i.e. all possible outcomes. $\mathcal{F}$ is the event space, i.e. a σ-algebra of subsets of Ω. A σ-algebra is a non-empty set satisfying:

1) $\Omega \in \mathcal{F}$;
2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;
3) If $A_n \in \mathcal{F}, n = 1,2,\cdots$, then $\cup_n A_n \in \mathcal{F}$.

$\mathcal{P}: \mathcal{F} \to [0,1]$ is the probability function satisfying:

i) $P(\Omega) = 1$ (normed);
ii) If $A_n \in \mathcal{F}$ $(n = 1,2,\cdots)$ is a countable collection of pairwise disjoint sets, then $\mathcal{P}(\cup_n A_n) = \sum_n \mathcal{P}(A_n)$ (σ-additive) .

Note 1. A set $A \in \mathcal{F}$ is called an event or $\mathcal{F}$-measurable.

Note 2. The Borel σ-algebra $\mathcal{B}$ is defined as the smallest σ-algebra in $\mathbb{R}$ that includes all intervals of real numbers. A set $B \in \mathcal{B}$ is called a Borel set.

Filtrations

Definition 2. Let $I = [0, T]$ with $T > 0$, and $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. A family $\{\mathcal{F}_t\}_{t\in I}$ of sub-σ-algebras of $\mathcal{F}$ is called a filtration associated with the probability space, if $\mathcal{F}_s \subset \mathcal{F}_t$, whenever $s \leq t$.

Example 1. Throw a fair coin three times.

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. A filtration can be constructed as follows.

1) In the time when no toss is performed, $\mathcal{F}_0 = \{\emptyset, \Omega\}$;
2) In the time when the first toss is done: $A_H = \{HHH, HHT, HTH, HTT\}$ and $A_T = \{THH, THT, TTH, TTT\}$, $\mathcal{F}_1 = \{\emptyset, \Omega, A_H, A_T\}$.

3) In the time when the first two tosses are done: $A_{HH} = \{HHH, HHT\}$, $A_{HT} = \{HTH, HTT\}$, $A_{TH} = \{THH, THT\}$, and $A_{TT} = \{TTH, TTT\}$, $\mathcal{F}_2 = \{\emptyset, \Omega, A_H, A_T, A_{HH}, A_{HT}, A_{TH}, A_{TT}, A_{HH}^c, A_{HT}^c, A_{TH}^c, A_{TT}^c, A_{HH} \cup A_{HT}, \cdots\}$.

4) In the time when all three tosses are done, $\mathcal{F}_3 = \mathcal{F}$.

Basically we do not want to use the total information $\mathcal{F}$. But we wish to use filtered shorter information $\mathcal{F}_t$ from the past to the present time t.

For a stochastic process $X(t)$, we often choose the natural filtration $\mathcal{F}_t = \sigma(X_s; 0 \le s \le t)$ of the σ-algebras generated by the stochastic process up to (and including) time t, which contains the past and present 'information' of the process.

Definition 3. A stochastic process $\{X_t\}_{t\in I}$ is adapted to a filtration $\{\mathcal{F}_t\}_{t\in I}$ or is $\mathcal{F}_t$-adapted if, for each $t \in I$, X_t is $\mathcal{F}_t$-measurable.

Note 3. A stochastic process $\{X_t\}_{t\in I}$ is always adapted to its natural filtration, which is the smallest filtration to which it is adapted.

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Definition 4. Given a stochastic process $\{X_t\}_{t\in I}$ on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a filtration $\{\mathcal{F}_t\}_{t\in I}$ in that space, we say that the stochastic process is an $\mathcal{F}_t$-martingale if,

1) It is adapted to the filtration $\{\mathcal{F}_t\}_{t\in I}$;

2) $X_t \in L^1$ (i.e. $E[|X_t|] < +\infty$), for $\forall t \in I$;

3) $E[X_t|\mathcal{F}_s] = X_s$ for any $s \le t$.

Note 4. A stochastic process $(Y_t)_{t\ge 0}$ is a martingale with its natural filtration, if for all $t \ge 0$,

1) $E(Y_t|Y_r, 0 \le r \le s) = Y_s$, for all $0 \le s \le t$.

2) $E(|Y_t|) < \infty$.

Note 5. A most important property of martingales is that they have constant expectation, since

$$E[X_n] = \int_\Omega X_n\, dp = \int_\Omega E\,(X_{n+1}|\mathcal{F}_n) dp = \int_\Omega X_{n+1}\, dp = E[X_{n+1}]$$

Example 2. The simple symmetric random walk is a martingale.

Solution Let

$$X_i = \begin{cases} +1, & \text{with probability} 1/2, \\ -1, & \text{with probability} 1/2, \end{cases} \quad i = 1,2,\cdots.$$

For $n \ge 1$, let $S_n = X_1 + X_2 + \cdots + X_n$, with $S_0 = 0$.

Since X_{n+1} is independent of $S_0, \cdots, S_n$, $E(X_{n+1}|S_0, \cdots, S_n) = E(X_{n+1})$. Since $E(g(X)|X) = g(X)$, $E(S_n|S_0, \cdots, S_n) = S_n$. Therefore,

$$\begin{aligned} E(S_{n+1}|S_0, \cdots, S_n) &= E(X_{n+1} + S_n|S_0, \cdots, S_n) \\ &= E(X_{n+1}|S_0, \cdots, S_n) + E(S_n|S_0, \cdots, S_n) \\ &= E(X_{n+1}) + S_n = S_n. \end{aligned}$$

The second part of the martingale definition is satisfied as

$$E(|S_n|) = E\left(|\sum_{i=1}^{n} X_i|\right) \leq E\left(\sum_{i=1}^{n} |X_i|\right) = \sum_{i=1}^{n} E(|X_i|) = n < \infty.$$