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# Calculating reactor transfer functions by Padé approximation via Lanczos algorithm

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**Abstract**

An effective method of solving the one-speed transport equation in frequency domain is demonstrated in this paper, the so-called Padé approximation via Lanczos algorithm (PVL). The advantage of the PVL method is that implementing the calculation process over a considerably reduced model yields a pseudo-analytical expression of the transfer function over a fairly large range of frequency. As a particular application, the dynamic transfer function of a reactor, i.e. the neutron noise induced by a localised perturbation is calculated in one-speed transport theory. The problem is essentially the same as that of the "detector-field-of-view"; studied by other authors. The PVL algorithm is demonstrated through the solution of the problem and its advantages are described. The quantitative results show that although one-speed theory was used, a local component was found, and thus the local-global decomposition could be reconstructed. This shows that unlike in diffusion theory where at least two-group theory is necessary, the local behaviour can be described already by a one-speed equation in transport theory. 

## 1. Introduction

The primary purpose of this paper was to test an effective algorithm for the solution of a transport problem. This algorithm is called the "Padé approximation via Lanczos algorithm" or PVL algorithm (Feldmann and Freund, 1995). The algorithm

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is described in detail in the paper. The starting point is a finite difference discrete ordinate scheme for the solution of the transport equation, which in the traditional form would lead to a matrix equation of high dimensions. The high dimensionality makes explicit methods impractical or indeed impossible to apply, and the application of model reduction technique becomes necessary. The PVL method treats the problem with a series expansion of the inverse of a matrix operator, and after once calculating the individual terms of this expansion by operation on matrices of a significantly lower dimension, the full information of the transfer function in a fairly wide frequency range can be recovered. This way the numerical load of the algorithm is considerably decreased.

As an application of the algorithm, we selected a case in reactor noise. Due to the complexity of the dynamic problem, unlike for static cases, most problems of reactor noise theory are treated in a one-group or two-group diffusion approach (Behsinger et al., 1977; Dam, 1977; Hagen et al., 1992; Kosály, 1975). There are only a few reported cases of noise calculations with transport theory (Analytis, 1980, 1983; Kosály and Sanchez, 1985). Analytis gave a solution in an infinite medium with analytical methods, whereas Kosály and Sanchez treated a transport problem numerically. In both cases the authors used multigroup theory, and, not surprisingly, found the existence of the local component.

It is, however, natural to expect that the local component of the noise should exist already in one-speed transport theory. It is well-known that in this approach the flux in an infinite subcritical system by a point source contains an asymptotic and a transient term (Bell and Glasstone, 1970). The asymptotic part is very similar to the solution with one-group diffusion theory, whereas the transient term is localised around the source. It is also known that there is a close similarity between the equations for the static flux induced by a point source in a subcritical system and that of the noise induced by a point perturbation in a critical system. Based on this analogy alone, the expectation that local behaviour should be seen already in one-speed transport theory is justified.

One can also argue in an alternative way. The local component of neutron noise was found first in the classic paper by Weinberg and Schweinler (1948). The treatment used there was diffusion theory with a slowing-down kernel. Because of the slowing-down kernel, such a theory is analogous to the transport equation in that it is not a pure differential equation, rather an integro-differential one. As was also discussed in Pázsit (1981), continuous slowing down theory is equivalent with multigroup theory with an infinite number of groups, and many-group and even two-group theory are simplifications of the continuous slowing down case. As soon as there are at least two groups, a local component occurs whose characters do not change much with the introduction of more groups, or even with the transition to slowing-down theory. So there are natural similarities between the extension of the one-group diffusion theory to many-group diffusion theory on one hand, and to one-speed transport theory (i.e. an integro-differential equation) on the other.

The numerical results obtained in this work, which are also used for the illustration of the PVL algorithm, clearly show the appearance of a local component. Similarly to the local component found in two-group diffusion theory, the local term

found here depends on the frequency much weaker than the global term, and dominates at higher frequencies. Moreover, it appears at lower frequencies in larger systems rather than in small systems, completely in accordance with expectations based on physical grounds attributed to the local component known from earlier works.

## 2. Description of the physical problem

Consider a homogeneous critical bare slab reactor of thickness $2a$ Under the assumption of isotropic scattering, the neutron balance can be described by the following transport equation (Bell and Glasstone, 1970):

$$\mu \frac{\partial}{\partial x}\phi(x,\mu) + \Sigma_t \phi(x,\mu) = \frac{c\Sigma_t}{2}\int_{-1}^{1} \phi(x,\mu')\mathrm{d}\mu' \tag{1}$$

$$\phi(\pm a, \mp\mu) = 0, \quad 1 > \mu \geqslant 0 \tag{2}$$

Here $\Sigma_t = \Sigma_a + \Sigma_s$ is the total cross-section, and $c$ is the average number of secondaries, as usual.

We shall now calculate the variations of the signal of a point detector due to localised fluctuations in the absorption cross-section. That is, the variations in the signal of the detector are assumed to arise from small changes of the macroscopic absorption cross-section:

$$\Sigma_a \;\Rightarrow\; \Sigma_a + \gamma\delta\big(x - x_p\big)\delta\Sigma_a(t) \tag{3}$$

where $\gamma$ is the so-called Galanin's constant. $x_p$ is the perturbation position. For simplicity of notations, we assume that $\gamma \equiv 1$ in the following.

The changes result in small fluctuations of the following quantities:

$$\phi(x,\mu) \;\Rightarrow\; \phi(x,\mu) + \delta\phi(x,\mu,t) \tag{4}$$

$$C(x) \;\Rightarrow\; C(x) + \delta C(x,t) \tag{5}$$

$$c \;\Rightarrow\; c(t) = c + \delta c(t) \tag{6}$$

where the last relationship stands for the fluctuation of the criticality factor.

As usual we shall use linear theory which is obtained by neglecting second-order terms, i.e. neglection of the product of fluctuating quantities. This also means that in the time-dependent case, on the right hand side of (1), (and also in the corresponding expression for the delayed neutron precursors), to the first order, $c(t)\Sigma_t(t)$ can be replaced by $c\Sigma_t$. Hence the kinetic equations for the fluctuations can be written as:

$$\frac{1}{v}\frac{\partial}{\partial t}\delta\phi(\mathrm{x},\mu,t) = -\mu\frac{\partial}{\partial x}\delta\phi(\mathrm{x},\mu,t) - \Sigma_t\delta\phi(\mathrm{x},\mu,t) + \frac{c(1-\beta)\Sigma_t}{2}\int_{-1}^{1}\delta\phi(x,\mu',t)\mathrm{d}\mu' + \lambda\delta C(x,t) - \delta\big(x-x_{\mathrm{p}}\big)\delta\Sigma_{\mathrm{a}}(t)\phi(x,\mu) \tag{7}$$

$$\frac{\partial}{\partial t}\delta C(x,t) = \frac{c\beta\Sigma_t}{2}\int_{-1}^{1}\delta\phi(x,\mu',t)\mathrm{d}\mu' - \lambda\delta C(x,t) \tag{8}$$

$$\delta\phi(\pm a,\mp\mu,t) = 0, \quad 1 > \mu \geqslant 0 \tag{9}$$

We shall calculate the neutron noise in the reactor, which will be a function of both the perturbation position $x_{\mathrm{p}}$ and the detection coordinate $x_{\mathrm{D}}$. Formally (van Dam, 1976), one can assume that there is a point detector located at $x = x_{\mathrm{D}}$ represented by

$$\Sigma_{\mathrm{D}}(x) = \Sigma_{\mathrm{D}}\delta(x - x_{\mathrm{D}}) \tag{10}$$

where $\Sigma_{\mathrm{D}}$ is the effective absorption cross-section of the detector, dimensionsless in this 1-D model. Then the detector response $\delta R$ to the source $-\delta\big(x-x_{\mathrm{p}}\big)\times\delta\Sigma_{\mathrm{a}}\,(t)\phi(x,\mu)$ can be written as

$$\delta R(x_{\mathrm{D}},x_{\mathrm{P}},t) = \int_{-a}^{a}\int_{-1}^{1}\Sigma_{\mathrm{D}}(x)\delta\phi(x,\mu,t)\mathrm{d}x\mathrm{d}\mu, \tag{11}$$

or

$$\delta R\big(x_{\mathrm{D}},x_{\mathrm{p}},t\big) = \Sigma_D\int_{-1}^{1}\delta\phi(x_{\mathrm{D}},\mu,t)\mathrm{d}\mu. \tag{12}$$

For simplicity of notations, we assume that $\Sigma_{\mathrm{D}} \equiv 1$ in the following.

As usual, the time derivative and the equation for the delayed neutron precursors are eliminated by temporal Fourier transform of Eqs. (7) and (8). Setting

$$\delta\phi(x,\mu,t\ \rightarrow\ -\infty) = 0; \quad \delta C(x,t\ \rightarrow\ -\infty) = 0,$$

the Fourier transform

$$F(s) = \int_{-\infty}^{\infty}\mathrm{e}^{-st}f(t)\mathrm{d}t \tag{13}$$

of Eqs. (7) and (8) leads to

$$s\delta\phi(x,\mu,s) = -v\mu\frac{\partial}{\partial x}\delta\phi(x,\mu,s) - v\Sigma_t\delta\phi(x,\mu,\mathrm{s}) + \frac{c(1-\beta)v\Sigma_t}{2}\int_{-1}^{1}\delta\phi(x,\mu',s)\mathrm{d}\mu' + v\lambda\delta C(x,s) - v\delta\big(x-x_{\mathrm{p}}\big)\delta\Sigma_a(s)\phi(x,\mu) \tag{14}$$

$$s\delta C(x,\mathrm{s}) = \frac{c\beta\Sigma_t}{2}\int_{-1}^{1}\delta\phi(x,\mu',\mathrm{s})\mathrm{d}\mu' - \lambda\delta C(x,s) \tag{15}$$

where $s = i\omega$. The boundary conditions become

$$\delta\phi(\pm a, \mp\mu, s) = 0, \quad 1 > \mu \geqslant 0 \tag{16}$$

The response characteristics of the detector are then summarized in terms of the transfer function:

$$G\left(x_{\mathrm{D}}, x_{\mathrm{p}}, s\right) = \frac{\delta R\left(x_{\mathrm{D}}, x_{\mathrm{p}}, s\right)}{\delta\Sigma_a(s)} \tag{17}$$

To a constant, this is equivalent to the Green's function (transfer function) of the system. Keeping $x_{\mathrm{p}}$ constant, the dependence of $G\left(x_{\mathrm{D}}, x_{\mathrm{p}}, s\right)$ on $x_{\mathrm{D}}$ yields the detector field-of-view. As mentioned in the introduction, this function has been so far investigated in the framework of different homogeneous multi-group diffusion-theory and transport-theory models. In the transport theory calculations, an infinite or a semi-infinite system was assumed, but no finite systems such as a slab with two free boundaries were investigated. As mentioned by Zweifel (1967), no closed form solutions to (14) involving two interfaces has been obtained yet, thus in general practical numerical solutions have been sought instead. In the method presented here we obtain a quasi-analytical solution.

From the physical point of view, one novelty of the present paper is the one-speed transport theory treatment of the noise equations and the existence of a local component. It will be seen that one-group homogeneous transport theory is sufficient to reconstruct the two components (local and global) of the detector response.

From the numerical point of view, the algorithm has a considerable advantage when it comes to calculations including frequency dependence. In noise calculations, the transfer function often needs to be calculated for several frequencies between 2 and 100 Hz, which may be rather time-consuming with traditional methods. We are going to demonstrate that the full information of the transfer function or detector field-of-view can be recovered through solving Eq. (14) for one fixed value of the $s(s = i\omega)$ by use of the Padé approximation via Lanczos algorithm.

## 3. Discrete mathematical model

The calculational procedure starts with a discretisation with respect to the spatial and the angular variables, i.e. use a finite difference discrete ordinates method as described in Bell and Glasstone (1970). In order to use symmetry to reduce the size of the discrete mathematical model, we suppose that the perturbation takes place at the centre of the system, i.e. $x_{\mathrm{p}} = 0$. This restriction is naturally not necessary and can be dropped easily, but our purpose here is only demonstration and this is why

we treat a simplified case. Notation on $x_p$ will be dropped in the continuation and $x_D$ will be re-denoted as $x$. Due to symmetry, the following relationships hold:

$$\delta\phi(-x, \mu, s) = \delta\phi(x, -\mu, s)$$
$$\delta C(x, s) = \delta C(-x, s) \tag{18}$$

The fact that the boundary conditions are specified over half of the range of the angular variable $\mu$ will be made use of when choosing the form of discretisation, see (23) and (24).

To avoid handling a Dirac-delta function explicitly in Eq. (3), we use the following representation of Dirac-delta function (Morse and Feshbach, 1953)

$$\delta(x) = \lim_{\zeta \to 0} \Delta(\zeta, x) \tag{19}$$

where $\Delta(\zeta, x) = \frac{1}{\pi}\left[\frac{\zeta}{\zeta^2 + x^2}\right]$.

Concretely, we handled the following point-like perturbation

$$\Delta(x) = \frac{1}{\pi}\left[\frac{10^{-8}}{10^{-16} + x^2}\right] \tag{20}$$

Because of the symmetry properties mentioned above, we discretize the integral by choosing an even-order Gauss quadrature set of directions $\{\mu_j\}$ and weights, $\{\omega_j\}$, j = 1, 2, ..., 2N, such that

$$\mu_j = -\mu_{2N+1-j} \quad \text{and} \quad \omega_j = \omega_{2N+1-j} \tag{21}$$

Then we discretize the spatial variable $x$ by choosing the $2K+2$ mesh points as

$$-a = x_0 < x_1 < \ldots < x_K < 0, \quad h_k = x_k - x_{k-1}, \quad k = 1, \ldots, K \tag{22}$$

and $0 < -x_K \ldots < -x_1 < -x_0 = a$.

The derivative terms are then approximated by finite difference such as

$$\frac{\partial}{\partial x}\delta\phi(x, \mu_j, s)|_{x=x_{k+\frac{1}{2}}} \approx \frac{\delta\phi(x_{k+1}, \mu_j, s) - \delta\phi(x_k, \mu_j, s)}{h_{k+1}} \tag{23}$$

and

$$\delta\phi\left(x_{k+\frac{1}{2}}, \mu_j, s\right) \approx \frac{\delta\phi(x_{k+1}, \mu_j, s) + \delta\phi(x_k, \mu_j, s)}{2} \tag{24}$$

where $x_{k+\frac{1}{2}} = \frac{x_{k+1}+x_k}{2}$. The same approximate method is used for $\delta C(x, s)$. Then we introduce a number of notations as follows:

$$\tau = \frac{s+\lambda}{2}, \quad c_1 = -\frac{\nu c(1-\beta)\Sigma_t}{4}; \quad c_2 = -\frac{c\beta\Sigma_t}{4}; \quad \lambda_1 = \frac{\nu\Sigma_t - \lambda}{2};$$

$$\delta\phi_{kj} = \delta\phi\big(x_k, \mu_j, s\big); \delta C_k = \delta C(x_k, s);$$

$$^{\phi}(k+1/2)j = \frac{\phi\big(x_k, \mu_j\big)\Delta(\zeta, x_k) + \phi\big(x_{k+1}, \mu_j\big)\Delta(\zeta, x_{k+1});}{2}$$

$$\delta\Phi_j = \big(\delta\phi_{1j}, \ldots, \delta\phi_{(k+1)j}\big)^T; \quad \delta C = \big(\delta C_1, \ldots, \delta C_{(k+1)}\big)^T; \;\; ;$$

$$\Phi_j = -\nu\Big(\phi_{\frac{1}{2}j} \ldots, \phi_{(K+\frac{1}{2})j}\Big)^T; \quad \underset{k}{\mathbf{l}_j = (0, 0, \ldots, \omega_j, \ldots, \omega_j, \ldots, 0, 0)^T;}_{K+1-k}$$

$$\mathbf{X} = (\delta\Phi_1, \ldots, \delta\Phi_N, \delta C)^T; \;\; ;$$

$$\mathbf{b} = (\Phi_1, \ldots, \Phi_N, 0)^T; \mathbf{l} = (l_1, \ldots, l_N, 0)^T;$$

$$M = \begin{bmatrix} M1 & & & \\ & M1 & & \\ & & \ldots & \\ & & & M1 \end{bmatrix}; \qquad M1 = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ \ldots & & & & & \\ 0 & 0 & 0 & \ldots & 1 & 1 \end{bmatrix};$$

$$M2_j = \begin{bmatrix} \frac{\nu\mu_j}{h_1} & 0 & 0 & \ldots & 0 & 0 \\ -\frac{\nu\mu_j}{h_2} & \frac{\nu\mu_j}{h_2} & 0 & \ldots & 0 & 0 \\ 0 & -\frac{\nu\mu_j}{h_3} & \frac{\nu\mu_j}{h_3} & \ldots & 0 & 0 \\ \ldots & & & & & \\ 0 & 0 & 0 & \ldots & -\frac{\nu\mu_j}{h_{K+1}} & \frac{\nu\mu_j}{h_{K+1}} \end{bmatrix};$$

$$W1_j = \begin{bmatrix} \omega_j & 0 & 0 & \ldots & 0 & 0 \\ \omega_j & \omega_j & 0 & \ldots & 0 & 0 \\ 0 & \omega_j & \omega_j & \ldots & 0 & 0 \\ \ldots & \ldots & & & & \\ 0 & 0 & 0 & \ldots & \omega_j & \omega_j \end{bmatrix}; \qquad W2_j = \begin{bmatrix} 0 & 0 & \ldots & 0 & \omega_j & \omega_j \\ 0 & 0 & \ldots & \omega_j & \omega_j & 0 \\ \ldots & \ldots & & & & \\ \omega_j & \omega_j & \ldots & 0 & 0 & 0 \\ \omega_j & 0 & \ldots & \ldots & 0 & 0 \end{bmatrix};$$

$$H = H1 + H2$$

$$H1 = \begin{bmatrix} M2_1 + \lambda_1 M1 & & & \frac{-v\lambda}{2} M1 \\ & & & \frac{-v\lambda}{2} M1 \\ M2_2 + \lambda_1 M1 & \dots & & \\ & & M2_N + \lambda_1 M1 & \frac{-v\lambda}{2} M1 \\ & & 0 & 0 \end{bmatrix};$$

$$H2 = \begin{bmatrix} c_1(W1_1 + W2_1) & c_1(W1_2 + W2_2) & \dots & c_1(W1_N + W2_N) & 0 \\ c_1(W1_1 + W2_1) & c_1(W1_2 + W2_2) & \dots & c_1(W1_N + W2_N) & 0 \\ \dots & & & & \\ c_1(W1_1 + W2_1) & c_1(W1_2 + W2_2) & \dots & c_1(W1_N + W2_N) & 0 \\ c_2(W1_1 + W2_1) & c_2(W1_2 + W2_2) & \dots & c_2(W1_N + W2_N) & 0 \end{bmatrix},$$

With the above notations, the discrete problem can be written as

$$\tau MX = -HX + \mathbf{b}\delta\Sigma_a \tag{25}$$

$$\delta \mathrm{R} = \mathbf{l}^T \mathbf{X} \tag{26}$$

Thus the transfer function Eq. (17) becomes

$$G(\tau) = \mathbf{l}^T (H + \tau M)^{-1} \mathbf{b} \tag{27}$$

Assuming that $M^{-1}H$ is diagonalizable, or, $M^{-1}H = S\Lambda S^{-1}, \Lambda = \mathrm{diag}\left(\lambda_1, \dots, \lambda_{(N+1)(2K+1)}\right)$, where $\lambda_1$ are the eigenvalues of $M^{-1}H$ and the columns of $S$ are the corresponding eigenvectors, we obtain

$$G(\tau) = \mathbf{l}^T \left(M^{-1}H + \tau\right)^{-1} M^{-1} b = \underbrace{\mathbf{l}^T S}_{=y^T} (\Lambda + \tau)^{-1} \underbrace{S^{-1} M^{-1} \mathbf{b}}_{=z} = \sum_{i=1}^{(N+1)(2K+1)} \frac{y_i z_i}{\tau + \lambda_i} \tag{28}$$

where $y_i$ and $z_i$ are the components of the vectors $\mathbf{y}$ and $\mathbf{z}$.

However the numerical computation of all eigenvectors and eigenvalues of the matrix $M^{-1}H$ becomes prohibitively expensive as soon as its size reaches a few hundreds, which is the case in all practical situations. Therefore the only practical way of solution is through an approximation. In the following we use the Padé approximation via Lanczos algorithm (PVL) (Feldmann and Freund, 1995) to evaluate Eq. (27).

## 4. Padé approximation

Let $\tau_0$ be an arbitrary fixed complex number such that $H + \tau_0 M$ is non-singular. Using a shift of variables as $\tau = \alpha + \tau_0$ and setting

$$A = -(H + \tau_0 M)^{-1} M; \quad \mathbf{r} = (\tau_0 M)^{-1} \mathbf{b}; \tag{29}$$

we can rewrite Eq. (27) as

$$G(\tau_0 + \alpha) = \mathbf{l}^T (I - \alpha A)^{-1} \mathbf{r} \tag{30}$$

If the spectral radius $\gamma(\alpha A)$ is less than unity, the Neumann series expansion can be used to get

$$G(\tau_0 + \alpha) = \mathbf{l}^T \big(I + \alpha A + \alpha^2 A^2 + \ldots\big)\mathbf{r} = \sum_{i=0}^{\infty} m_i \alpha^i, \tag{31}$$

where $m_i = \mathbf{l}^T A^i r, \quad i = 0, 1, \ldots,$ are called the moments of the frequency-response function . The purpose is to truncate the expansion (31) after a relatively low number of terms.

The truncation $G_q$ of $2q$ terms of Eq. (31) is called by the $q$th Padé approximation. In order to implement the $q$th Padé approximation, the task is to compute the leading $2q$ moments $m_0, m_1, \ldots, m_{2q-1}$ of $G$. Eq. (31) itself can be used to calculate the moments by the recursive solution of the linear systems

$$(H + \tau_0 M) u_k = -M u_{k-1} \tag{32}$$

with the initial vector

$$(H + \tau_0 M) u_0 = b, \tag{33}$$

where $u_0 = \mathbf{r}, u_1 = A\mathbf{r}, \ldots, u_{2q-1} = A^{2q-1}\mathbf{r}$. This is the so-called asymptotic waveform evaluation algorithm (AWE) based on Padé approximation (Gallivan et al., 1994). In AWE, the severe numerical problem is that $u_k = A^k \mathbf{r}$ converges quickly to the eigenvector associated with the eigenvalue with the largest absolute value and thus only the information corresponding to this eigenvalue can be used. On the other hand, in view of (28), the function $G(\tau)$ to be approximated clearly depends on all the eigenvalues of A. Therefore, in order to recover information about more than one eigenvalue, a better algorithm or the PVL algorithm was proposed in Feldmann and Freund (1995). The basic idea can be summarized as follows.

First, one calculates a tridiagonal matrix $T_q$ via the Lanczos process from such that

$$A V_q = V_q T_q + \big[0, \ldots, 0, v_{q+1}\big] \rho_{q+1}, \tag{34}$$

where $V_q = [v_1, \ldots, v_{q-1}, v_q]$ is a matrix of dimension $(N+1)(2K+1) \times q$ and, $T_q$ is a tridiagonal matrix of dimension $q$, and $\rho_{q+1}$ is a number, and $v_{q+1}$ is a vector of size $(N+1)(2K+1)$.

From (34), one can show that

Table 1
Basic data used for our calculation, v = 2200 m/s, $\beta = 0.00625$, $\lambda = 0.0767\ s^{-1}$

| | Theoretical model | Practical model |
|---|---|---|
| Thickness of the slab 2a (cm) | 6 | 200 |
| Total cross section $\Sigma_t$ (1/cm) | 1 | 0.50 |
| Neutron secondaries per collision | 1.4025 | 1.0003 |

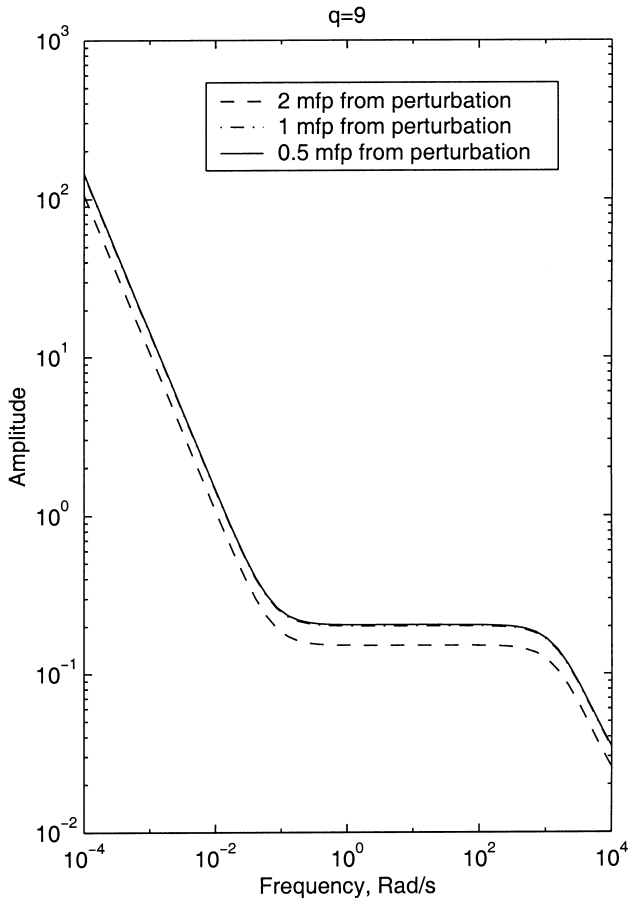


Fig. 1. Amplitudes of the transfer functions for a 6 mfp thickness slab reactor while the detector is at three different positions.

$$m_i = \left(\mathbf{l}^T\mathbf{r}\right)\left(\mathrm{e}_1^T T_q^i e_1\right), \quad i = 0, 1, \ldots, 2q-1., \tag{35}$$

where, $e_1 = (1, 0, \ldots, 0)^T \in R^q$, and thus

$$G_q(\tau_0 + \alpha) = \sum_{i=0}^{2q} m_i \alpha^i = \left(l^T r\right)\sum_{i=0}^{2q} e_1^T \left(\alpha T_q\right)^i e_1 = \left(\mathbf{l}^T\mathbf{r}\right)\mathrm{e}_1^T\left(1 - \alpha T_q\right)^{-1} e_1 \tag{36}$$

Then using the eigen decomposition

$$T_q = S_q \mathrm{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_q\right) S_q^{-1} \tag{37}$$

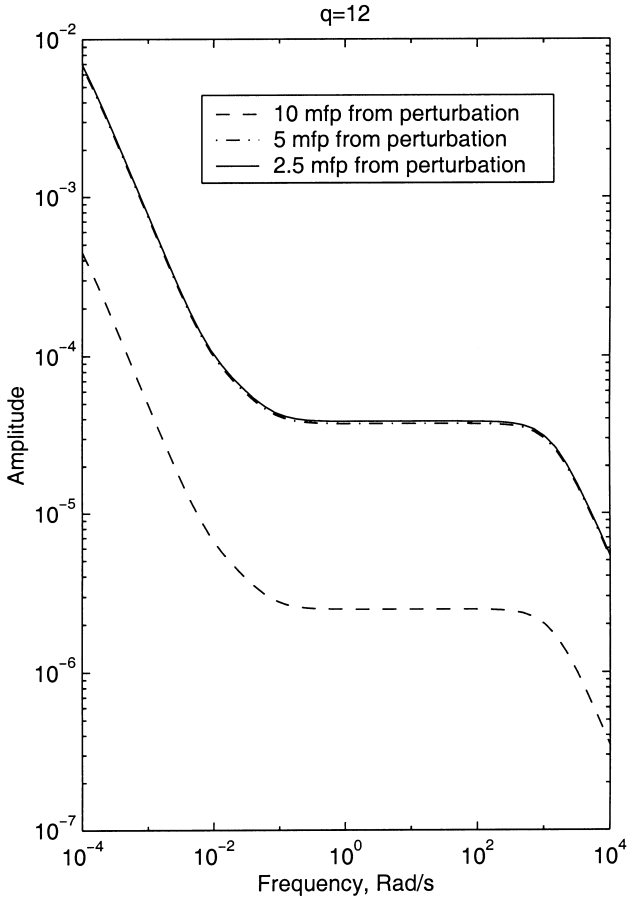


Fig. 2. Amplitudes of the transfer functions for a 200 cm thickness slab reactor while the detector is at three different positions.

of $T_q$, and setting

$$\mu = S_q^T e_1 \quad and \quad \mathrm{v} = S_q^{-1} e_1 \quad , \tag{38}$$

one can derive the Padé approximant $G_q$ from (36) as:

$$G_q(\tau_0 + \alpha) = \left(\mathbf{l}^T \mathbf{r}\right) \sum_{j=1}^{q} \frac{\mu_j v_j}{1 - \alpha \lambda_j} \tag{39}$$

Apparently the complexity of the calculation has been simplified considerably as compared to the direct solution of (27) because the size of $T_q$ is much smaller than A as we will see in the next Section where numerical results will be given. The more important thing is that we only need to implement the calculation process once for a fixed shift $\tau_0$, after which we can obtain the pseudo-analytical expression of the transfer function $G$.

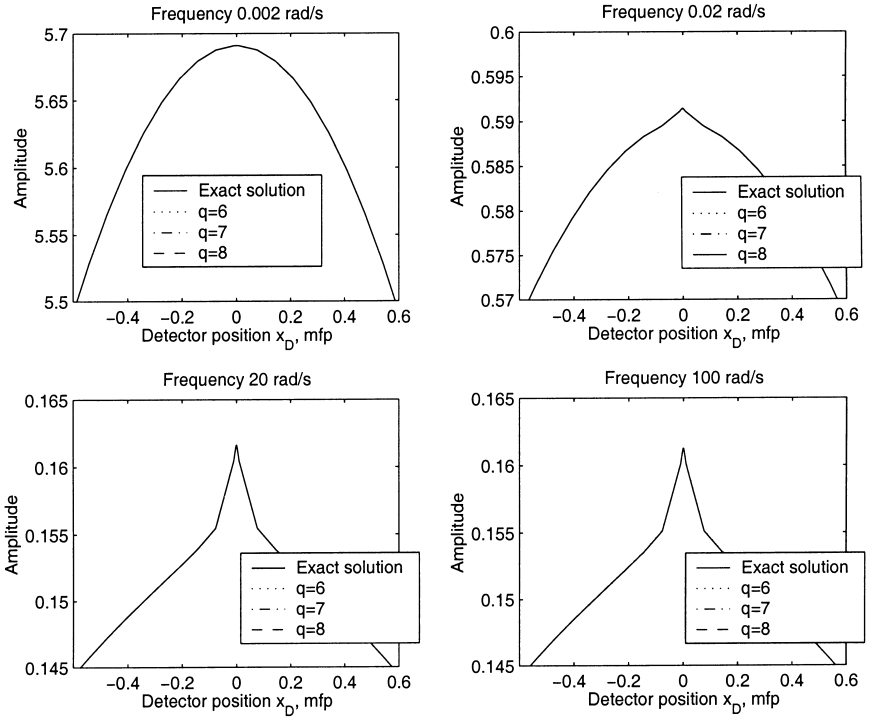


Fig. 3. The local-global decomposition of the noise induced by a central perturbation at various frequencies for a 6 mfp thickness reactor.

## 5. Numerical results

The numerical computation was performed for two different critical systems. One is a small system, with size six mean free paths, and the other being a "realistic" model with system size equal to 100 mfp. The data used were listed in Table 1. For the small system, a total of $2K+2=102$ mesh points, and $N=24$ positive directions were used. Thus the matrix size is $(N+1)(2K+1)=2525$. For the large system, a total of $2K+2=212$ mesh points, and $N=14$ positive directions were used. Thus the matrix size is $(N+1)(2K+1)=3165$. In both cases, two step lengths were used. Near the local perturbation, the step length about 0.001 mfp was chosen. In both cases, the shift $\tau_0 = 0.01$ was used.

The calculated transfer functions by use of (39) are shown in Figs. 1 and 2 for the small system and the large system, respectively, as functions of the frequency. The frequency behaviour of the absolute value of the transfer function is similar to that calculated in diffusion theory, i.e. it diverges at low frequencies, is constant at the so-called plateau frequencies, and decreases again above the plateau frequency.

It is maybe somewhat more illuminating to investigate the space dependence of the transfer function for a few selected frequencies. The space dependence of the

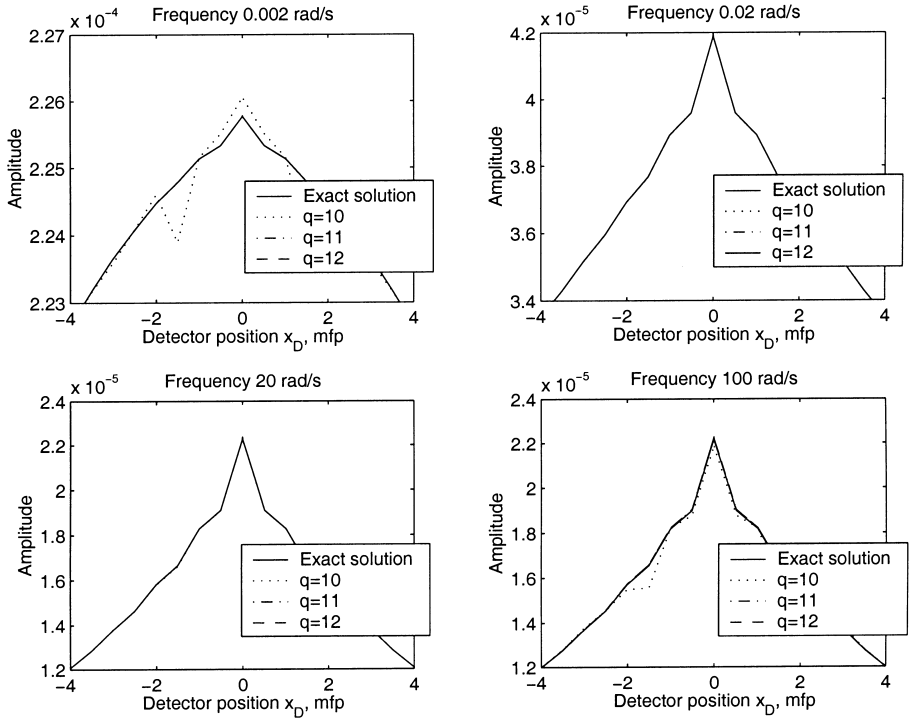


Fig. 4. The local-global decomposition of the noise induced by a central perturbation at various frequencies for a 200 cm thickness reactor.

noise induced by the perturbation (i.e. the transfer function or detector field-of-view G) was calculated at four different frequencies by the use of (25)–(27) directly (we shall call it the exact solution) and (39), respectively. The comparison is shown in Figs. 3 and 4 for the small system and the large system, respectively, in the close vicinity of the perturbation. As the figures show, the approximate solutions fit the exact solutions very well. The saving in the numerical work is obvious if we compare it with the dimensionality of the matrix problem listed above.

The spatial behaviour of the solutions shows a definite resemblance to two-group diffusion theory results that have been obtained in the past. One can clearly discern a global and a local component. The global component is point kinetic at the lowest frequency, and starts to deviate from the point kinetic behaviour with increasing frequencies. This is in a very good agreement with earlier results. In addition, however, a local component occurs. This component appears to be independent of the frequency, in contrast to the global component. At the lowest frequency, the local component is suppressed by the global one, but it gradually becomes more and more dominating with increasing frequencies. This behaviour is completely in accordance with the properties of the local-global decomposition as known from two-group diffusion theory.

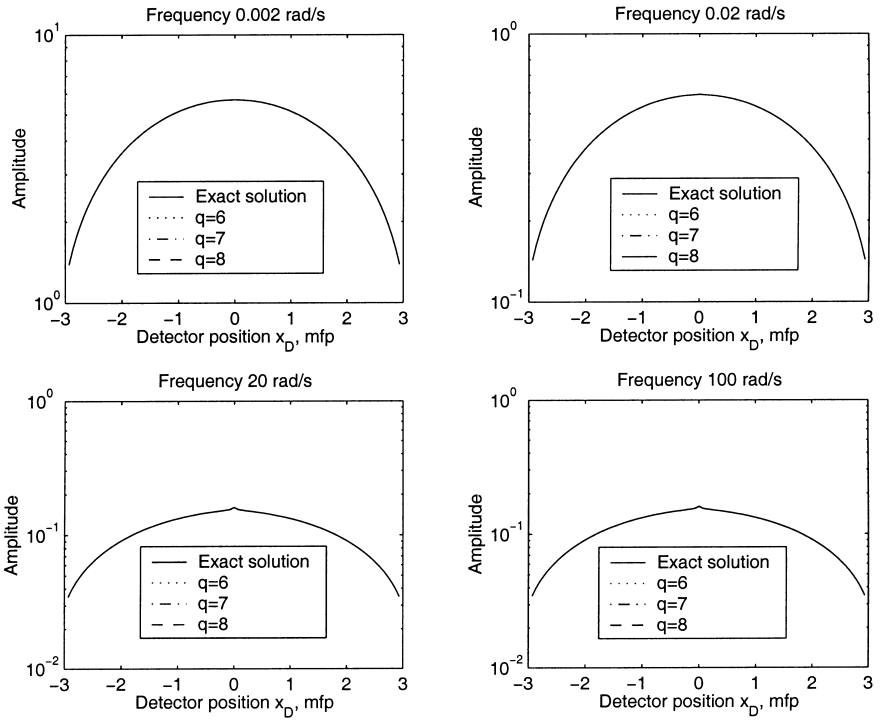


Fig. 5. The existence of the boundary layer of the noise at various frequencies for a 6 mfp thickness slab reactor.

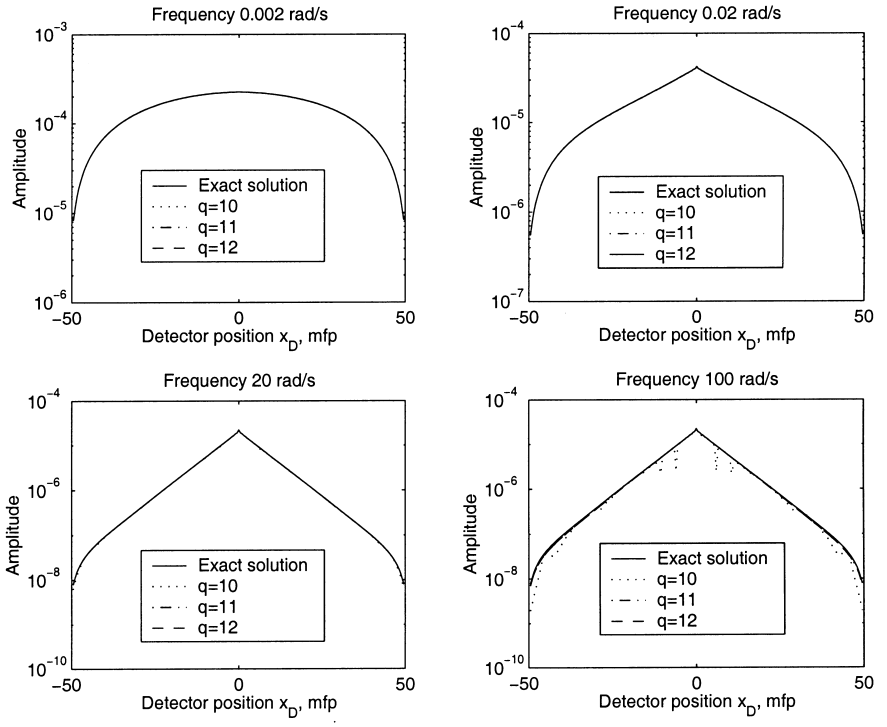


Fig. 6. The existence of the boundary layer of the noise at various frequencies for a 100 mfp thickness slab reactor.

The behaviour of the global component is also compatible with expectations and previous experience. In the larger system the space dependence of the global component deviates from point kinetics already at low frequencies. At high frequency, it becomes quite localised around the perturbation.

Apart from the appearance of the local component, there is one more property of the present transport theory solution which is absent in the diffusion theory solutions. It concerns the existence of a boundary layer close to the system boundary. This can be seen in Figs. 5 and 6, which show the detector-field-of view of the two systems in logarithmic scale. One can easily see the occurrence of a boundary layer in the solution, especially in the larger system. This phenomenon is well known in static calculations, but has not yet been shown in dynamic transport theory calculations.

## 6. Discussion and conclusions

The main purpose of this paper originally was to test and demonstrate the performance of a numerical scheme, the Padé approximation via Lanczos algorithm (PVL) method in solving a transport problem. The numerical results show that the

PVL algorithm has excellent efficiency and stability and thus it can be used for solving certain transport problems.

However, the concrete application which we selected has yielded some interesting new results in space-dependent reactor dynamics. Namely, the noise induced by a localised perturbation of variable strength, which is equivalent with the Green's function of the general noise problem, was calculated in one-group transport theory. It was found that the solution obtained shows properties which agree with experiment, but whose reconstruction requires an energy-dependent treatment, at least two-group theory, if diffusion theory is used. Our solution contains both a global and a local noise component. As is well-known, the local component cannot be obtained in one-group diffusion theory. Our results show that a local component can be obtained already at the level of one-speed transport theory.

The deficiency of the one-group diffusion theory in yielding a local component is a result of the fact that in n-group theory there are $n$ spatial eigenvalues (relaxation constants). Thus in 1-group theory there is only one spatial eigenvalue, which is the static buckling with a slow spatial relaxation. Two- and more group theory, and indeed also continuous slowing-down theory (Pázsit, 1981) yields, in addition to the fundamental mode, also higher spatial eigenvalues that are all much larger in absolute value than that of the global component.

In transport theory one obtains more than one spatial eigenvalues already in a one-group treatment. For example with a static source in an infinite medium one has one discrete eigenvalue which describes the asymptotic behaviour, and a continuous spectrum which describes the transient behaviour close to the source (Bell and Glasstone, 1970). It is the largest value of the discretised form of this continuous spectrum which we found in our numerical work and which corresponds to the local component.

The transport theory solution in addition also exhibits a boundary layer. This is another physical novelty of the solution presented here. For the description of a boundary layer, it is not sufficient to use two-group diffusion theory, due to the obvious difficulties of diffusion theory to treat free surfaces. Two-group diffusion theory can reconstruct a "boundary layer" at the boundary of the core in a reflected reactor (the reflector peak) but not the boundary layer at a free surface. The transport solution given here, one the other hand, exhibits also the boundary layer.

It is of course and interesting question how the local component of one-group transport theory and the local component of two-group diffusion theory are related to each other. It was shown for instance in an earlier paper (Pázsit 1981) that the local and global components of two-group diffusion theory and diffusion theory with a continuous slowing-down kernel are numerically very similar, but the origin of the local component had different interpretations in the two approaches. The relationship between two-group diffusion theory and one-group transport theory is the subject of further investigation and the results will be reported in future work.

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