The general backward theory of neutron fluctuations in subcritical systems with multiple emission sources

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Summary. — The Feynman-alpha and Rossi-alpha formulae are derived analytically for subcritical systems driven by a multiple emission source, *i.e.* one that emits several neutrons on each source emission event. The prime example of such sources is a spallation source, which will be used in future accelerator driven subcritical systems (ADS) such as the energy amplifier. Such formulae for ADS have been derived recently but in simpler neutronic models. In this paper six groups of delayed neutron precursors are taken into account, and the full joint statistics of the prompt and all delayed groups is included. Thus the present results are generalisations of earlier ones. The involved problems that are encountered are solved with a combination of effective analytical techniques and symbolic algebra codes.

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1. – Introduction

Fluctuations in the number of neutrons or in the number of detector counts in a subcritical reactor with a source have long been used to determine various reactor parameters [1,2]. Although in principle such methods are capable to determine both nuclear parameters and subcritical reactivity, it is the latter, *i.e.* measuring the subcritical reactivity that has been by far the most important application.

Two fluctuation-based methods, the Feynman-alpha or variance-to-mean, and the Rossi-alpha or covariance method have been used extensively. As their name already

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suggests, they are both based on the measurement of the second moment of the statistics of the detector counts. In the Feynman method, one determines the relative variance

$$\frac{\sigma_z^2(t)}{\langle Z(t)\rangle} \tag{1}$$

as a function of the measurement time t. Here, $Z(t)$ is the detector count in $(0,t)$, a random variable, $\langle Z(t)\rangle$ is its expected value and $\sigma_z^2(t)$ its variance. With the use of the theory of linear Markov processes (master equations), used also in this paper, the following expression, the so-called Feynman-alpha formula, has been derived for the variance-to-mean, refs. [3-5]:

$$\frac{\sigma_Z^2}{\langle Z\rangle} = 1 + \varepsilon \sum_{i=0}^{6} A_i \left(1 - \frac{1-e^{-\alpha_i t}}{\alpha_i t}\right). \tag{2}$$

In the above, ε is the detection efficiency. The most important part of the sum is the prompt part, *i.e.* $i = 0$, for which one has

$$\alpha_0 \approx \frac{\beta-\rho}{\Lambda}, \qquad \rho = \frac{k-1}{k}, \qquad \alpha_i \ll \alpha_0;\ i \geq 1 \tag{3}$$

and

$$A_0 = \frac{\langle \nu(\nu-1)\rangle}{\bar{\nu}^2}\frac{1}{(\beta-\rho)^2} \equiv \frac{D_\nu}{(\beta-\rho)^2}, \tag{4}$$

where β is the delayed neutron fraction, ρ the subcritical reactivity, Λ the prompt neutron generation time. The quantities $\bar{\nu}$, $\langle \nu(\nu-1)\rangle$ and D_ν are related to the various moments of the number distribution of the fission neutrons; for their definitions we refer to the Nomenclature in the appendix. For simplicity, we drop the bar from $\bar{\nu}$ in the following text. We only note here that eq. (2) is derived with the assumption of a Poisson source, *i.e.* the probability of emitting one neutron in $\mathrm{d}t$ is $S\,\mathrm{d}t$.

The physical interpretation of (2) is as follows. If all neutrons were statistically independent, as, *e.g.,* those emitted by a radioactive source, the statistics would be Poisson and the relative variance equal to unity. However, in a multiplying medium, each neutron will induce a chain, leading to the generation of a total on $\frac{1}{1-k}$ neutrons in an infinite system. All neutrons in such a chain are correlated due to the fact that they have a common ancestor. Due to positive correlations, the variance will be higher than Poisson. It is this part of the variance-to-mean which exceeds unity, where the useful information on the system is found. However, each individual chain will die out in a subcritical reactor, the die-out being determined by the time constants α_i; this is why the relative variance saturates.

The Rossi-alpha method is based on the measurement of the covariance function of the detector counts in infinitesimal time intervals $\mathrm{d}t$ around times t and $t+\tau$, defined as

$$P(\tau)\,\mathrm{d}\tau = \frac{C_{zz}(\tau)}{\langle Z\rangle} = \frac{\langle Z(t+\tau)Z(t)\rangle - \langle Z(t+\tau)\rangle\langle Z(t)\rangle}{\langle Z\rangle}. \tag{5}$$

With similar techniques as in the foregoing case (master equations) an analytical expression for the Rossi-alpha formula can be derived in the form

$$P(\tau)\,\mathrm{d}\tau = \varepsilon\,\mathrm{d}\tau \sum_{i=0}^{6} C_i e^{-\alpha_i \tau}. \tag{6}$$

Here the time constants α_i are the same as in (2) and (3), and the constants C_i similar to A_i in (4). The interpretation of (6) is also similar to that of the Feynman-alpha formula: if all neutrons were independent, the covariance would be zero. Due to the branching (fission) and corresponding generating of correlations, the covariance is larger than zero, but it dies out exponentially with the same time constants as the ones that appear in the Feynman formula.

The theory behind such fluctuation processes received a renewed interest recently with the advent of the so-called accelerator driven systems (ADS). Such systems, intended to be used either for energy production or transuranium transmutation, will use a subcritical core with a strong spallation source [6-13]. Since such systems will be run in a subcritical mode, it will be important to measure, and continuously monitor, the subcriticality of the system. Based on the experience with traditional systems, the Feynman- and Rossi-alpha methods appear as suitable candidates for such purposes.

However, there is an important difference between the traditional systems and a future ADS. Namely, the neutron source in an ADS, usually assumed to be a spallation source, has statistical properties that are different from those of the traditionally used radioactive sources. Radioactive sources have simple Poisson statistics, where all source neutrons are independent. In the derivation of the Feynman- and Rossi-alpha formulae for traditional systems, this simple source statistics was assumed. In a spallation source, all neutrons arising from the spallation reactions of one primary projectile, usually a proton, are correlated, and thus the source statistics is not Poisson, rather it is a compound Poisson distribution [14]. Although the spallation neutrons give rise to by one projectile are generated in an intra- or inter-nuclear cascade, and are thus born within a finite time span and not simultaneously, this time span is very short (a few ns) compared to the generation time of fission neutrons in the fission chain, and is even shorter than the lifetime of the prompt neutron chain [15, 16]. The arrival times of the projectile (*e.g.,* a proton) are assumed to obey Poisson statistics with a time constant that is much larger than the cascade generation time in the target. Thus, to a good approximation, one can assume that the source neutrons obey a compound Poisson statistics, at least in conceptual studies like the present one. In other words, it means that there will exist correlations between the neutrons that were generated, in contrast to the traditional case, not in the fission chain but in the external source. It is necessary to investigate the effect of such correlations on the Feynman- and Rossi-alpha formulas.

The formal task is therefore to re-derive the Feynman- and Rossi-alpha formulas with the assumption of compound Poisson source statistics. This has actually been done recently by several authors [17-22]. However, in all works at most one average delayed neutron group was assumed (in some papers the delayed neutrons were not even explicitly taken into account). The recent work consists in the generalisation of the method to the case when six different delayed neutron groups are distinguished. Besides the value of the results themselves, some novel mathematical methods were employed in this paper in order to arrive at the present results. First, analytical methods were used to derive the solution of the single-particle–induced distributions in a compact form. Second, a symbolic algebra code (Mathematica) was used to evaluate the integral of these distributions to arrive to the moments of the source-induced cascade. We also believe that the results have a general reference value. The fact is that details of the original derivations of the Feynman- and Rossi-alpha formulae are not available in journal publications or in monographs. Also, the explicit exact values of the various factors appearing in the Feynman- and Rossi-alpha formulae are not available in the most common references such as [2], only simplified (approximate) values. This paper contains the most complete list of formulae with an explicit use of six delayed neutron groups as well as full prompt-prompt, prompt-delayed and delayed-delayed correlations and the multiplicity of the source.

2. – General principles

As usual in all works that aim to obtain analytical solutions, the space and energy dependence of the neutronic model will be kept simple. We shall use an infinite homogeneous reactor which permits the use of point kinetics, *i.e.* a space-independent theory. An energy-independent, *i.e.* one-group model will be used. The limitations of the latter for spallation-source–induced neutron fluctuations were discussed in [20]. The detector is assumed to be spatially homogeneous and infinite and its finite volume will be conceptually simulated by a detector efficiency factor. On the other hand, the nuclear model of the fission chain will be quite detailed. Not only the correlations between prompt neutrons, but prompt-delayed and delayed-delayed correlations will also be allowed. With all likelihood, these latter correlations are negligible in realistic cases, and thus they were neglected in most of the previous work. Here we treat them explicitly. The other aspect, mentioned above, is the fact that 6 delayed neutron groups will be taken into account.

Both the source and the medium are assumed to be stationary, thus the arising fluctuations in the number of neutrons and in the detector counts will be stationary too. This means that the expected value and the variance of the number of neutrons in the system is constant, that of the detector counts between t and $t-T$ is independent of t, and the joint expected value of having one count in an infinitesimal time around t and one around $t+\tau$ depends only on τ. Regarding the source statistics, stationarity means that the distribution of source emission events is a Poisson process with constant (time-independent) parameter. That is, the probability of one source emission in $(t, \mathrm{d}t)$ is equal to

$$S\,\mathrm{d}t \tag{7}$$

where S is constant in both space and time. On each emission a random number of neutrons is released with a probability distribution

$$p_q(n); \qquad \sum_{n=0}^{\infty} p_q(n) = 1. \tag{8}$$

Calculation of the above moments and correlations leading to the Feynman and Rossi formulae goes in two stages. First, the statistics of the single-particle–induced cascade need to be determined. Since we will only be dealing with second moments at most, we need to calculate at most the second-order or two-point distribution

$$P\big(N_1, Z_1, t; N_2, Z_2, t+\tau\big), \tag{9}$$

which is the probability of having N_1 neutrons in the system at time t and N_2 neutrons at time $t+\tau$, and having had Z_1 and Z_2 detector counts in some time intervals T or $\mathrm{d}t$ preceding t and $t+\tau$, respectively, due to one single initial neutron at time zero. As we shall see soon, in order to derive a master equation for this distribution, the delayed neutrons must also accounted for explicitly. However, at this point we only want to relate the single-particle– and source–induced distributions, respectively, in which relationship the delayed neutron precursors do not appear. Hence only the above simpler distribution needs to be considered. For obvious reasons, the distribution (9) is not stationary, *i.e.* it depends on both t and τ separately. For its generating function a backward-type (or rather a mixed but predominantly backward-type) master equation will be derived and solved for the first two moments both in the one-time (Feynman-alpha) and the two-time (Rossi-alpha) case.

In the second step, from this distribution, the source-induced stationary distribution needs to be determined. This is achieved by the generalised Bartlett formula, which gives a relationship between the single-particle–induced distributions (9) and the stationary source-induced distribution

$$\tilde{P}\big(N_1, Z_1, t; N_2, Z_2, t+\tau\big). \tag{10}$$

The interpretation of this distribution is the same as that of (9) above, with the difference that the cascade was induced by switching on the source at $t=0$ with the further initial condition of having no neutrons in the system for $t \le 0$. The stationary distribution and its moments can be determined by calculating the limit of (10) for $t \to \infty$.

Introducing the generating functions G and $\tilde{G}$ as

$$G\big(x_1, z_1, t; x_2, z_2, t+\tau\big) = \sum_{N_1}\sum_{Z_1}\sum_{N_2}\sum_{Z_2} x_1^{N_1} z_1^{Z_1} x_2^{N_2} z_2^{Z_2} P\big(N_1, Z_1, t; N_2, Z_2, t+\tau\big) \tag{11}$$

and similarly for $\tilde{G}$, the Bartlett formula for multiple emission sources in the stationary (*i.e.* asymptotic) case can be written as [20]

$$\begin{aligned} \tilde{G}\big(x_1, z_1, x_2, z_2, \tau\big) &= \lim_{t\to\infty} \tilde{G}\big(x_1, z_1, t; x_2, z_2, t+\tau\big) = \\ &= \exp\left[S\int_0^\infty \mathrm{d}t\left[\sum_n p_q(n) G^n\big(x_1, z_1, t; x_2, z_2, t+\tau\big) - 1\right]\right]. \end{aligned} \tag{12}$$

This equation will be used below when one- and two-point second moments of the stationary distributions will be calculated. It will be used to relate the first and second moments of the source-induced distribution to those of the single-particle–induced distributions. These relationships express the source-induced moments as integrals over the single-particle–induced moments as (12) suggests and will be given here as follows. Defining the asymptotic mean value of the source-induced distribution as

$$\tilde{N} \equiv \lim_{t\to\infty}\langle \tilde{N}(t)\rangle = \frac{\partial}{\partial x_1}\tilde{G}\big(x_1, z_1, x_2, z_2, \tau\big)\big|_{x_1=z_1=x_2=z_2=1}, \tag{13}$$

the derivation of (12) with respect to x_1 yields

$$\tilde{N} = \bar{q} S \int_0^\infty N(t)\,\mathrm{d}t, \tag{14}$$

where

$$\bar{q} \equiv \sum_n n p_q(n). \tag{15}$$

We also used the short-hand notation

$$N(t) \equiv \langle N(t) \rangle. \tag{16}$$

In general, in the equations when first-order moments are given explicitly as in (13), we shall omit the sign of the expected value since there is no risk to mix up the expected value with the corresponding random variable.

Similarly, for the first moment of the stationary detector count one obtains

$$\tilde{Z} = \bar{q} S \int_0^\infty Z(t)\,\mathrm{d}t. \tag{17}$$

Equations (14) and (17) can be evaluated after the single-particle–induced expected values $N(t)$ and $Z(t)$ are determined. This will be performed in the next section. The single-particle–induced mean value $N(t)$ will play a central role since it will be shown to serve as a Green's function to several of the other moments.

The asymptotic values of the one-point second moment of the source-induced neutron number or detector count can be derived as follows. First, define

$$M_{NN}(t) \equiv \langle N(t)[N(t)-1] \rangle = \frac{\partial^2}{\partial x_1^2} G\big(x_1, z_1, t; x_2, z_2, t+\tau\big)\big|_{x_1=z_1=x_2=z_2=1} \tag{18}$$

and

$$\tilde{M}_{NN} \equiv \big\langle \tilde{N}\big(\tilde{N}-1\big) \big\rangle = \frac{\partial^2}{\partial x_1^2} \tilde{G}\big(x_1, z_1, x_2, z_2, \tau\big)\big|_{x_1=z_1=x_2=z_2=1}. \tag{19}$$

Then, from (12), with twofold derivation, one obtains for the modified second moment of the asymptotic source-induced distribution, defined as

$$\tilde{\mu}_{NN} \equiv \tilde{M}_{NN} - \tilde{N}^2, \tag{20}$$

the expression

$$\tilde{\mu}_{NN} = \bar{q} S \int_0^\infty M_{NN}(t)\,\mathrm{d}t + \langle q(q-1) \rangle S \int_0^\infty N^2(t)\,\mathrm{d}t, \tag{21}$$

where

$$\langle q(q-1) \rangle = \sum_n n(n-1) p_q(n). \tag{22}$$

Equation (21) will be later simplified when it will be shown that the single-particle–induced second moment M_{NN} can be expressed as a convolution over a source function with $N(t)$ as a Green's function.

From (12) one obtains a similar expression for the detector counts as

$$\tilde{\mu}_{ZZ} = \bar{q}S\int_0^\infty M_{ZZ}(t)\,\mathrm{d}t + \langle q(q-1)\rangle S\int_0^\infty Z^2(t)\,\mathrm{d}t, \tag{23}$$

where $\tilde{\mu}_{ZZ}$ and $M_{ZZ}(t)$ are defined similarly to (20) and (18). Again, this will be simplified by expressing $M_{ZZ}(t)$ with a convolution integral.

The asymptotic two-point second moments, *i.e.* the stationary values of the covariance or correlation function of the neutron number at t and $t+\tau$, and that of the detector counts around t and $t+\tau$, are defined as

$$\begin{aligned}\tilde{C}_{NN}(\tau) &= \lim_{t\to\infty}\big\langle\tilde{N}(t)\tilde{N}(t+\tau)\big\rangle - \big\langle\tilde{N}(t)\big\rangle\big\langle\tilde{N}(t+\tau)\big\rangle = \\ &= \frac{\partial^2}{\partial x_1\partial x_2}\tilde{G}\big(x_1,z_1,x_2,z_2,\tau\big)\big|_{x_1=z_1=x_2=z_2=1} - \tilde{N}^2.\end{aligned} \tag{24}$$

A simple derivation of (12) results in

$$\tilde{C}_{NN}(\tau) = \bar{q}S\int_0^\infty M_{NN}(t,\tau)\,\mathrm{d}t + \langle q(q-1)\rangle S\int_0^\infty N(t)N(t+\tau)\,\mathrm{d}t, \tag{25}$$

where

$$M_{NN}(t,\tau) \equiv \langle N(t)N(t+\tau)\rangle. \tag{26}$$

Similarly,

$$\tilde{C}_{ZZ}(\tau) = \bar{q}S\int_0^\infty M_{ZZ}(t,\tau)\,\mathrm{d}t + \langle q(q-1)\rangle S\int_0^\infty Z(t)Z(t+\tau)\,\mathrm{d}t. \tag{27}$$

In the next two sections, the single-particle–induced moments $N(t)$, $Z(t)$, $M_{NN}(t)$, $M_{ZZ}(t)$, $M_{NN}(t,\tau)$ and $M_{ZZ}(t,\tau)$ will be determined and the integrals (14), (17), (21), (23), (25) and (27) evaluated. It will be seen that the second-order moments $M_{NN}(t)$, etc. need not be calculated explicitly, only the corresponding source functions and the integrals will be performed over the source functions in (21), (23), (25) and (27).

3. – Derivation of the Feynman-alpha formula

We shall start with a master equation for the joint probability distribution of the number of neutrons and precursors at time t and the number of detector counts up to time t in the system. As mentioned earlier, when calculating the single-particle–induced distributions, it is necessary to consider the delayed neutrons explicitly, because they appear in fission. In the preceding section, dealing with the generalised Bartlett formula, we could disregard delayed neutrons because the Bartlett formula is derived from the source emission properties and the source does not emit delayed neutron precursors.

In this section we shall only deal with the Feynman-alpha formula and thus shall only use "one-point" distributions and their moments. We shall assume a stationary reactor, which means that the statistics at time t, induced by one initial particle at t_0, will only depend on $t-t_0$. This fact can be used to retain notation on one time instant only.

When deriving a backward equation based on the first instant method, it will actually be a “mixed”-type equation, because the time derivative will be taken with respect to the terminal (detection) time. However, the scattering operator will be a backward-type one, referring to the interactions of the initial particle, and in all other respects we shall utilise the properties of the backward formalism.

According to the above, we shall derive coupled equations for the following quantities. Let

$$P\big(N, C_1, C_2, \dots, C_6, Z, t\big) \tag{28}$$

be the probability that there are N neutrons and C_i precursors in the i-th group at time t in the system, induced by one initial neutron at $t = 0$, and that there have been Z detector counts between t and $t - T$. Likewise, let

$$Q_j\big(N, C_1, C_2, \dots, C_6, Z, t\big) \tag{29}$$

be the probability that there are N neutrons and C_i precursors in the i-th group at time t in the system, induced by one initial precursor in group j at $t = 0$, and that there have been Z detector counts between t and $t - T$. We define the corresponding probability generating functions as

$$\begin{aligned} G\big(x, y_1, y_2, \dots, y_6, z, t\big) &= \\ &= \sum_N \sum_{C_1} \sum_{C_2} \cdots \sum_{C_6} \sum_Z x^N y_1^{C_1} y_2^{C_2} \cdots y_6^{C_6} z^Z P\big(N, C_1, C_2, \dots, C_6, Z, t\big) \end{aligned} \tag{30}$$

and

$$\begin{aligned} H_j\big(x, y_1, y_2, \dots, y_6, z, t\big) &= \\ &= \sum_N \sum_{C_1} \sum_{C_2} \cdots \sum_{C_6} \sum_Z x^N y_1^{C_1} y_2^{C_2} \cdots y_6^{C_6} z^Z Q_j\big(N, C_1, C_2, \dots, C_6, Z, t\big). \end{aligned} \tag{31}$$

The initial conditions for the above quantities read as

$$P\big(N, C_1, C_2, \dots, C_6, Z, 0\big) = \delta_{N,1}\delta_{C_1,0}\delta_{C_2,0}\dots\delta_{C_6,0}\delta_{Z,0}, \tag{32}$$

$$G\big(x, y_1, y_2, \dots, y_6, z, 0\big) = x, \tag{33}$$

$$Q_j\big(N, C_1, C_2, \dots, C_6, Z, 0\big) = \delta_{N,0}\delta_{C_1,0}\dots\delta_{C_j,1}\dots\delta_{C_6,0}\delta_{Z,0}, \tag{34}$$

and

$$H_j\big(x, y_1, y_2, \dots, y_6, z, 0\big) = y_j. \tag{35}$$

For P a first-instant–type master equation can be obtained by adding the probabilities of the mutually exclusive possibilities of having no collision or one collision of the initial neutron within $(0, \mathrm{d}t)$, respectively. One then obtains

$$
\begin{aligned}
(36)\quad & P\big(N, C_1, C_2, \ldots, C_6, Z, t\big) = \\
&= (1 - \lambda_\mathrm{a}\,\mathrm{d}t) P\big(N, C_1, C_2, \ldots, C_6, Z, t - \mathrm{d}t\big) + \\
&\quad + \lambda_\mathrm{c}\,\mathrm{d}t\,\delta_{N,0}\delta_{C_1,0}\delta_{C_2,0}\cdots\delta_{C_6,0}\delta_{Z,0} + \lambda_\mathrm{f}\,\mathrm{d}t \sum_{l,m_j} p(l, m_1, m_2, \ldots, m_6) \times \\
&\quad \times \prod_{i=1}^{l} P\big(N^i, C_1^i, C_2^i, \ldots, C_6^i, Z^i, t\big) \prod_{j=1}^{6}\prod_{i=1}^{m_j} Q_j\big(N^{ji}, C_1^{ji}, C_2^{ji}, \ldots, C_6^{ji}, Z^{ji}, t\big) + \\
&\quad + \lambda_\mathrm{d}\,\mathrm{d}t\,\delta_{N,0}\delta_{C_1,0}\delta_{C_2,0}\cdots\delta_{C_6,0}\big[\Delta(t,T)\delta_{Z,1} + \bar{\Delta}(t,T)\delta_{Z,0}\big],
\end{aligned}
$$

where the function $\Delta(t, T)$ is defined as

$$
\Delta(t, T) = \begin{cases} 1, & \text{for } 0 \le t \le T, \\ 0, & \text{otherwise,} \end{cases} \tag{37}
$$

and

$$
\bar{\Delta}(t, T) = 1 - \Delta(t, T). \tag{38}
$$

The arguments of the products in eq. (36) are subject to the following constraints:

$$
\begin{gathered}
\sum_{i=1}^{l} N^i + \sum_{j=1}^{6}\sum_{i=1}^{m_j} N^{ji} = N, \\
\sum_{i=1}^{l} {C_k}^i + \sum_{j=1}^{6}\sum_{i=1}^{m_j} {C_k}^{ji} = C_k, \quad k = 1, \ldots, 6, \\
\sum_{i=1}^{l} Z^i + \sum_{j=1}^{6}\sum_{i=1}^{m_j} Z^{ji} = Z.
\end{gathered} \tag{39}
$$

With similar arguments, *i.e.* no decay or decay of the initial precursors, one obtains for the distributions Q_j

$$
\begin{aligned}
(40)\quad Q_j\big(N, C_1, C_2, \ldots, C_6, Z, t\big) &= \big(1 - \lambda_j\,\mathrm{d}t\big) Q_j\big(N, C_1, C_2, \ldots, C_6, Z, t - \mathrm{d}t\big) + \\
&\quad + \lambda_j\,\mathrm{d}t\, P\big(N, C_1, C_2, \ldots, C_6, Z, t\big).
\end{aligned}
$$

The symbols in the above equations have their usual meaning. A summary of notations used is given in the appendix.

From (36) and (40) we obtain for the generating functions G and H_j of (30) and (31) the following differential equations:

$$\begin{aligned}\frac{\partial G(x,y_1,y_2,\dots,y_6,z,t)}{\partial t} &= \lambda_{\mathrm{f}} \sum_{l,m_j} p(l,m_1,m_2,\dots,m_6) G^l(x,y_1,y_2,\dots,y_6,z,t) \times \\ &\times \prod_{j=1}^{6} H_j^{m_j}(x,y_1,y_2,\dots,y_6,z,t) + \lambda_{\mathrm{c}} - \\ &- \lambda_{\mathrm{a}} G(x,y_1,y_2,\dots,y_6,z,t) + \lambda_{\mathrm{d}}\{(z-1)\Delta(t,T)+1\}\end{aligned} \tag{41}$$

and

$$\begin{aligned}&\frac{\partial H_j(x,y_1,y_2,\dots,y_6,z,t)}{\partial t} = \\ &\qquad = \lambda_j\big\{G(x,y_1,y_2,\dots,y_6,z,t) - H_j(x,y_1,y_2,\dots,y_6,z,t)\big\}.\end{aligned} \tag{42}$$

The second of the above equations can be explicitly solved. By taking into account the initial condition (35) one gets

$$H_j\big(x,y_1,y_2,\dots,y_6,z,t\big) = \lambda_j \int_0^t e^{-\lambda_j(t-t')} G\big(x,y_1,y_2,\dots,y_6,z,t\big)\,\mathrm{d}t' + y_j e^{-\lambda_j t}. \tag{43}$$

Putting this back into (41) yields one single equation from which all statistics can be derived as

$$\begin{aligned}&\frac{\partial G\big(x,y_1,y_2,\dots,y_6,z,t\big)}{\partial t} = \\ &\quad = -\lambda_{\mathrm{a}} G\big(x,y_1,y_2,\dots,y_6,z,t\big) + \lambda_{\mathrm{d}}\big\{(z-1)\Delta(t,T)+1\big\} + \\ &\quad\quad + \lambda_{\mathrm{c}} + \lambda_{\mathrm{f}} \sum p\big(l,m_1,m_2,\dots,m_6\big) G^l\big(x,y_1,y_2,\dots,y_6,z,t\big) \times \\ &\quad\quad \times \prod_{j=1}^{6}\left(\lambda_j \int_0^t e^{-\lambda_j(t-t')} G\big(x,y_1,y_2,\dots,y_6,z,t'\big)\,\mathrm{d}t' + y_j e^{-\lambda_j t}\right)^{m_j}.\end{aligned} \tag{44}$$

Since the above equation does not contain any derivatives with respect to the transform variables x, y_i and z, for any moment, *i.e.* expected values of any order and any combination of variables, one single equation can be derived which can be solved separately from any other moment equations. The only technical difficulty of the solution is the calculation of certain integrals as will be seen soon. This is a difference compared to the forward equation where for the higher moments usually a coupled system of differential equations needs to be solved. The order of the system is increasing with the order of the moments. This, in general, constitutes more difficulties in the solution than the performing of the integrals in the backward case.

We shall now start deriving moments of this equation. For the first moment

$$N(t) \equiv \langle N(t)\rangle = \left.\frac{\partial G\big(x,y_1,y_2,\dots,y_6,z,t\big)}{\partial x}\right|_{x=y_i=z=1}, \quad i=1,\dots,6, \tag{45}$$

one obtains from the equation

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \lambda_\mathrm{f}\nu(1-\beta)N(t) + \lambda_\mathrm{f}\sum_{j=1}^{6}\nu\beta_j\lambda_j\int_0^t e^{-\lambda_j(t-t')}N(t')\,\mathrm{d}t' - \lambda_\mathrm{a}N(t) + \delta(t). \tag{46}$$

Here

$$\nu(1-\beta) \equiv \langle\nu_p\rangle = \sum_n\sum_{m_1}\cdots\sum_{m_6} np(n, m_1, \ldots, m_6) \tag{47}$$

and

$$\nu\beta_j \equiv \langle\nu_{d_j}\rangle = \sum_n\sum_{m_1}\cdots\sum_{m_6} m_j p(n, m_1, \ldots, m_j, \ldots, m_6). \tag{48}$$

In eq. (46) the initial condition (32) was added directly to the equation such that one has

$$N(t)|_{t=-0} = 0. \tag{49}$$

This was made to help realise later that the first moment $N(t)$ is the Green's function of the higher moments. A temporal Laplace transform of (46) yields

$$N(s) = \frac{\prod_{j=1}^{6}(s+\lambda_j)}{s^7 + a_1 s^6 + a_2 s^5 + \cdots + a_6 s + a_7}, \tag{50}$$

where the coefficients a_i, $i = 1, \ldots, 7$ are given by

$$\begin{aligned}
a_1 &= \sum_{j=1}^{6}\lambda_j + \frac{\beta-\rho}{\Lambda}, \\
a_2 &= \sum_{\substack{i,j=1\\ i<j}}^{6}\lambda_i\lambda_j + \sum_{j=1}^{6}\frac{\beta-\beta_j-\rho}{\Lambda}\lambda_j, \\
a_3 &= \sum_{\substack{i,j,k=1\\ i<j<k}}^{6}\lambda_i\lambda_j\lambda_k + \sum_{\substack{i,j=1\\ i<j}}^{6}\frac{\beta-(\beta_i+\beta_j)-\rho}{\Lambda}\lambda_i\lambda_j, \\
&\ldots \\
a_7 &= \frac{\beta-\left(\sum_{j=1}^{6}\beta_j\right)-\rho}{\Lambda}\prod_{j=1}^{6}\lambda_j = \frac{-\rho}{\Lambda}\prod_{j=1}^{6}\lambda_j.
\end{aligned} \tag{51}$$

Here the notation ρ and Λ were introduced as usual (see the appendix).

The inverse Laplace transform yields

$$N(t) = \sum_{i=0}^{6} z_i e^{s_i t}, \tag{52}$$

where s_0 and s_i, $i = 1, \ldots, 6$, are the seven roots of the denominator of (50), and

$$z_i = \frac{\prod_{j=1}^{6}(s_i + \lambda_j)}{\prod_{j \neq i}^{6}(s_i - s_j)}, \quad i = 0, 1, \ldots, 6. \tag{53}$$

Using (14), (50) and (51), one obtains for the stationary, source-induced mean value of neutrons

$$\tilde{N} = \bar{q} S N(s = 0) = \frac{\bar{q} S \Lambda}{-\rho}. \tag{54}$$

This is a known result that could have been derived directly from, *e.g.,* a deterministic point kinetic equation.

To calculate the mean and the modified variance of the detector counts of the source-induced cascade, we start with calculating from

$$Z(t,T) \equiv \langle Z(t,T) \rangle = \left. \frac{\partial G\big(x, y_1, y_2, \ldots, y_6, z, t\big)}{\partial z} \right|_{x=y_i=z=1}, \quad i = 1, \ldots, 6 \tag{55}$$

and

$$\begin{aligned} M_{ZZ}(t,T) &\equiv \langle Z(t,T)[Z(t,T) - 1] \rangle = \\ &= \left. \frac{\partial^2 G\big(x, y_1, y_2, \ldots, y_6, z, t\big)}{\partial z^2} \right|_{x=y_i=z=1}, \quad i = 1, \ldots, 6, \end{aligned} \tag{56}$$

respectively. For the first moment one obtains the equation

$$\begin{aligned} \frac{\mathrm{d}Z(t,T)}{\mathrm{d}t} &= \lambda_{\mathrm{f}} \nu (1 - \beta) Z(t,T) + \lambda_{\mathrm{f}} \sum_{j=1}^{6} \nu \beta_j \lambda_j \int_0^t e^{-\lambda_j (t-t')} Z(t',T)\, \mathrm{d}t' - \\ &- \lambda_{\mathrm{a}} Z(t,T) + \lambda_{\mathrm{d}} \Delta(t,T). \end{aligned} \tag{57}$$

The initial condition is $Z(0,T) = 0$. A comparison of (57) with eq. (46), accounting also for the initial condition for Z, shows that the first moment $N(t)$ is also the Green's function for $Z(t,T)$. Thus the solution can be written as

$$\begin{aligned} Z(t,T) &= \lambda_{\mathrm{d}} \int_0^t N(t-t') \Delta(t',T)\, \mathrm{d}t', \\ &= \begin{cases} \lambda_{\mathrm{d}} \displaystyle\int_0^t N(t-t')\, \mathrm{d}t' \equiv Z_1(t), & 0 \le t < T, \\ \lambda_{\mathrm{d}} \displaystyle\int_0^T N(t-t')\, \mathrm{d}t' \equiv Z_2(t,T), & t > T. \end{cases} \end{aligned} \tag{58}$$

From here, in the stationary case, one obtains by application of (17)

$$\tilde{Z}(T) = \bar{q} S \int_0^\infty Z(t,T)\, \mathrm{d}t = \varepsilon \lambda_{\mathrm{f}} \tilde{N} T, \tag{59}$$

which is of course the same result one obtains from forward theory [18].

The second moment can be calculated by applying (56) to (41) with the result

$$
\begin{aligned}
\frac{\mathrm{d}M_{ZZ}(t,T)}{\mathrm{d}t} &= \lambda_{\mathrm{f}}\nu(1-\beta)M_{ZZ}(t,T) + \\
&\quad + \lambda_{\mathrm{f}}\sum_{j=1}^{6}\nu\beta_j\lambda_j\int_0^t e^{-\lambda_j(t-t')}M_{ZZ}(t',T)\,\mathrm{d}t' - \\
&\quad - \lambda_{\mathrm{a}}M_{ZZ}(t,T) + Q_{ZZ}(t,T),
\end{aligned}
\tag{60}
$$

where the source term $Q_{ZZ}(t,T)$ is given by

$$
\begin{aligned}
Q_{ZZ}(t,T) = \lambda_{\mathrm{f}}\Bigg[&\langle\nu_p(\nu_p-1)\rangle Z^2(t,T) + \\
&+ 2\sum_{j=1}^{6}\langle\nu_p\nu_{d_j}\rangle Z(t,T)\lambda_j\int_0^t e^{-\lambda_j(t-t')}Z(t',T)\,\mathrm{d}t' + \\
&+ \sum_{j=1}^{6}\left\langle\nu_{d_j}\left(\nu_{d_j}-1\right)\right\rangle\left\{\lambda_j\int_0^t e^{-\lambda_j(t-t')}Z(t',T)\,\mathrm{d}t'\right\}^2 + \\
&+ 2\sum_{\substack{i,j=1\\ i<j}}^{6}\left\langle\nu_{d_i}\nu_{d_j}\right\rangle\left\{\lambda_i\int_0^t e^{-\lambda_i(t-t')}Z(t',T)\,\mathrm{d}t'\right\}\times \\
&\times\left\{\lambda_j\int_0^t e^{-\lambda_j(t-t')}Z(t',T)\,\mathrm{d}t'\right\}\Bigg].
\end{aligned}
\tag{61}
$$

Equation (60) has the same structure as (46), with the exception that the source term is different. The initial condition, similarly to that of M_{NN}, is

$$
M_{ZZ}(0) = 0,
\tag{62}
$$

see eq. (33). Since the source term in eq. (46) for $N(t)$ is a delta-function, as it was remarked earlier, this means that $M_{ZZ}(t,T)$ can be written as a convolution of the source term $Q_{ZZ}(t,T)$ with the Green's function $N(t)$ as

$$
M_{ZZ}(t,T) = \int_0^t Q_{ZZ}(t',T)N(t-t')\,\mathrm{d}t'.
\tag{63}
$$

Using (63) in (23) leads to

$$
\tilde{\mu}_{ZZ}(T) = \tilde{N}\int_0^\infty Q_{ZZ}(t,T)\,\mathrm{d}t + \langle q(q-1)\rangle S\int_0^\infty Z^2(t,T)\,\mathrm{d}t.
\tag{64}
$$

This expression will now be calculated. First, the integral can be further simplified. In fact, it is easy to show the following equalities by use of integration by parts:

$$(65)\quad \lambda_j \int_0^\infty \left\{ \int_0^t e^{-\lambda_j (t-t')} Z(t',T)\,\mathrm{d}t' \right\}^2 \mathrm{d}t = \\ = \int_0^\infty Z(t,T) \left(\int_0^t e^{-\lambda_j (t-t')} Z(t',T)\,\mathrm{d}t' \right) \mathrm{d}t$$

and

$$(66)\quad \int_0^\infty \left\{ \lambda_i \int_0^t e^{-\lambda_i (t-t')} Z(t',T)\,\mathrm{d}t' \right\} \left\{ \lambda_j \int_0^t e^{-\lambda_j (t-t')} Z(t',T)\,\mathrm{d}t' \right\} \mathrm{d}t = \\ = \frac{1}{\lambda_i + \lambda_j} \cdot \int_0^\infty \left\{ Z(t,T) \left(\int_0^t e^{-\lambda_i (t-t')} Z(t',T)\,\mathrm{d}t' \right) + \right. \\ \left. + Z(t,T) \left(\int_0^t e^{-\lambda_j (t-t')} Z(t',T)\,\mathrm{d}t' \right) \right\} \mathrm{d}t.$$

Moreover,

$$(67)\quad \Re(T) \equiv \int_0^\infty Z^2(t,T)\,\mathrm{d}t = \int_0^T \frac{\partial}{\partial x} \left(\int_0^\infty Z^2(t,x)\,\mathrm{d}t \right) \mathrm{d}x = \\ = 2 \int_0^T \int_0^\infty Z(t,x) \frac{\partial}{\partial x} Z(t,x)\,\mathrm{d}t\,\mathrm{d}x = \\ = 2\lambda_\mathrm{d} \int_0^T \int_0^\infty Z_2(t,x) N(t-x)\,\mathrm{d}t\,\mathrm{d}x,$$

where Z_2 is defined in (58), and

$$(68)\quad \varnothing_j(T) \equiv \\ \equiv \int_0^\infty Z(t,T) \left(\int_0^t e^{-\lambda_j (t-t')} Z(t',T)\,\mathrm{d}t' \right) \mathrm{d}t = \\ = \int_0^T \frac{\partial}{\partial x} \left\{ \int_0^\infty Z(t,x) \left(\int_0^t e^{-\lambda_j (t-t')} Z(t',x)\,\mathrm{d}t' \right) \mathrm{d}t \right\} \mathrm{d}x = \\ = \varepsilon \lambda_\mathrm{f} \int_0^T \left\{ \int_x^\infty e^{-\lambda_j t} \left[N(t-x) \left(\int_0^x e^{\lambda_j t'} Z_1(t')\,\mathrm{d}t' + \int_x^t e^{\lambda_j t'} Z_2(t',x)\,\mathrm{d}t' \right) + \right. \right. \\ \left. \left. + Z_2(t,x) \int_x^t e^{\lambda_j t'} N(t'-x)\,\mathrm{d}t' \right] \mathrm{d}t \right\} \mathrm{d}x.$$

Therefore (64) can be compacted as

$$\begin{aligned}\tilde{\mu}_{ZZ}(T) = {}& \big[\tilde{N}\lambda_{\mathrm{f}}\big\langle\nu_p\big(\nu_p-1\big)\big\rangle + \big\langle q(q-1)\big\rangle S\big]\Re(T) + \\ &+ \tilde{N}\lambda_{\mathrm{f}}\sum_{j=1}^{6}\lambda_j\big(2\big\langle\nu_p\nu_{d_j}\big\rangle + \big\langle\nu_{d_j}\big(\nu_{d_j-1}\big)\big\rangle\big)\varnothing_j(T) + \\ &+ 2\tilde{N}\lambda_{\mathrm{f}}\sum_{\substack{i,j=1\\ i<j}}^{6}\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}\big\langle\nu_{d_i}\nu_{d_j}\big\rangle\big(\varnothing_i(T)+\varnothing_j(T)\big).\end{aligned} \tag{69}$$

We can now write down an explicit expression for the Feynman-alpha formula. It is usually written in the form

$$Y(t) \equiv \frac{\tilde{\sigma}^2_{ZZ}}{\tilde{Z}} - 1 = \frac{\tilde{\mu}_{ZZ}(t)}{\tilde{Z}(t)}, \tag{70}$$

where $\tilde{Z}(t)$ and $\tilde{\mu}_{ZZ}(t)$ are given by (59) and (64), respectively, with re-denoting T as t. Using (61) and (69), it is only necessary to calculate $\Re(T)$ and $\varnothing_j(T)$. This was evaluated with Mathematica and manually.

Introducing the notations

$$\omega_i \equiv \frac{z_i}{s_i}\sum_{j=0}^{6}\frac{z_j}{s_i+s_j}, \quad i=0,1,\dots,6, \tag{71}$$

$$f_i(t) \equiv \left(1+\frac{1-e^{s_i t}}{s_i t}\right) = \left(1-\frac{1-e^{-\alpha_i t}}{\alpha_i t}\right), \quad i=0,1,\dots,6, \tag{72}$$

and noticing that

$$\sum_{j=0}^{6}\frac{z_j}{\lambda_i+s_j} = 0, \quad i=1,\dots,6, \tag{73}$$

we have the result

$$\begin{aligned}\frac{\tilde{\mu}_{ZZ}(t)}{\tilde{Z}(t)} = {}& 2\varepsilon\lambda_{\mathrm{f}}^2\left[\big\langle\nu_p\big(\nu_p-1\big)\big\rangle + \langle q(q-1)\rangle\frac{\nu}{\bar{q}}(-\rho)\right]\sum_{i=0}^{6}\omega_i f_i(t) - \\ &- 2\varepsilon\lambda_{\mathrm{f}}^2\sum_{j=1}^{6}\lambda_j^2\big(2\big\langle\nu_p\nu_{d_j}\big\rangle + \big\langle\nu_{d_j}\big(\nu_{d_j-1}\big)\big\rangle\big)\sum_{i=0}^{6}\frac{\omega_i f_i(t)}{s_i^2-\lambda_j^2} - \\ &- 4\varepsilon\lambda_{\mathrm{f}}^2\sum_{\substack{i,j=1\\ i<j}}^{6}\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}\big\langle\nu_{d_i}\nu_{d_j}\big\rangle\left(\lambda_i\sum_{k=0}^{6}\frac{\omega_k f_k(t)}{s_k^2-\lambda_i^2} + \lambda_j\sum_{k=0}^{6}\frac{\omega_k f_k(t)}{s_k^2-\lambda_j^2}\right).\end{aligned} \tag{74}$$

In (72), to facilitate comparison with the literature, the notations $\alpha_i \equiv -s_i$ were introduced.

For the system of one group of precursors, which was used in [21], one has

$$z_0 = \frac{s_0 + \lambda_1}{s_0 - s_1}, \qquad z_1 = -\frac{s_1 + \lambda_1}{s_0 - s_1}, \quad z_i = 0,\ \lambda_i = 0,\ i = 2, \ldots, 6. \tag{75}$$

With these relationships eq. (74) reduces to the formula (52) in [21] which is thus a special case of our result.

Equation (74) shows explicitly the effect of the multiple source on the Feynman-alpha formula. As was mentioned already in the previous papers using more simplified models, the effect of the multiple source is a modification of the amplitude of the term due to prompt correlations, whereas the time-dependence of the formula remains unchanged. This means that the evaluation method does not need to be changed. Due to the enhanced amplitude, the performance of the method is expected to be better in ADS than in traditional systems, especially with deep subcriticalities.

4. – Derivation of the Rossi-alpha formula

The derivation of the equations leading to the Rossi formula is very much the same as in the previous section, with the obvious difference that we have to keep track of two time instants. Moreover, the inhomogeneous or source term of the equation, expressing the detection process, will also be different. However the cascade model is the same and thus the equation will remain very similar to that of the previous section.

Again, we shall assume a stationary system, such that the probabilities at time t_1 and t_2, induced by one initial particle at t_0, will only depend on $t_1 - t_0$ and $t_2 - t_0$, respectively. Thus, similarly to the previous section, we shall set $t_0 = 0$ and $t_1 = t$, and, in addition, $t_2 = t + \tau$.

The variable corresponding to the detection time will correspond to short time intervals that can be regarded as infinitesimal. Due to this, very simple relationships will exist between the first and second moments of the neutron numbers at t and $t+\tau$ on the one hand, and the detector counts within $t - \mathrm{d}t, t$ and $t + \tau - \mathrm{d}t, t + \tau$ on the other hand. However, the second moments will obey different initial conditions for the two, and thus the moments will also differ, even if the difference is simple.

Hence we define the neutron- and precursor-induced two-point distributions, analogously to (28) and (29), as

$$P\big(N_1, C_{11}, C_{12}, \ldots, C_{16}, Z_1, t; N_2, C_{21}, C_{22}, \ldots, C_{26}, Z_2, t + \tau\big) \tag{76}$$

and

$$Q_j\big(N_1, C_{11}, C_{12}, \ldots, C_{16}, Z_1, t; N_2, C_{21}, C_{22}, \ldots, C_{26}, Z_2, t + \tau\big). \tag{77}$$

The corresponding generating functions, *i.e.*

$$G\big(x_1, y_{11}, y_{12}, \ldots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \ldots, y_{26}, z_2, t + \tau\big) \tag{78}$$

and

$$H_j\big(x_1, y_{11}, y_{12}, \ldots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \ldots, y_{26}, z_2, t + \tau\big), \tag{79}$$

are defined analogously to (30) and (31). The initial conditions for P, Q_j, and G and H_j can also be easily given as follows:

$$
\begin{aligned}
(80)\qquad & P\big(N_1, C_{11}, C_{12}, \dots, C_{16}, Z_1, 0; N_2, C_{21}, C_{22}, \dots, C_{26}, Z_2, \tau\big) = \\
&\qquad = \delta_{N_1,1}\delta_{C_{11},0}\cdots\delta_{C_{16},0}\delta_{Z_1,0}P\big(N_2, C_{21}, C_{22}, \dots, C_{26}, Z_2, \tau\big); \\
& G\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, 0; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, \tau\big) = \\
&\qquad = x_1 G\big(x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, \tau\big).
\end{aligned}
$$

Here the one-point generating function $G(x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t')$ is defined as

$$
\begin{aligned}
(81)\quad & G\big(x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t'\big) = \\
&\quad = G\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, 0; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t'\big)\big|_{x_1=y_{11}=y_{12}=\dots=y_{16}=z_1=1}
\end{aligned}
$$

and is obviously equivalent to the one-point distribution defined in the previous section. In a similar way, one will have

$$
\begin{aligned}
(82)\qquad & H_j\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, 0; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, \tau\big) = \\
&\qquad = y_{1j} H_j\big(x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, \tau\big), \quad j = 1, \dots, 6.
\end{aligned}
$$

It is easy to see that for calculating the correlations, the above initial conditions are sufficient.

As mentioned above, the structure of the equation is exactly the same as in the preceding section, thus we turn immediately to the equation for the generating functions. We will have

$$
\begin{aligned}
(83)\qquad & \frac{\mathrm{d}}{\mathrm{d}t} G\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t+\tau\big) = \\
&\quad = \lambda_{\mathrm{c}} - \lambda_{\mathrm{a}} G\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t+\tau\big) + \\
&\quad + \lambda_{\mathrm{d}}\big(z_1\Delta(t, \mathrm{d}t)c + z_2\Delta(t+\tau, \mathrm{d}t) + \bar{\Delta}_{12}\big) + \\
&\quad + \lambda_{\mathrm{f}} \sum_{l, m_j} p\big(l, m_1, m_2, \dots, m_6\big) \times \\
&\qquad\quad \times G^l\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t+\tau\big) \times \\
&\quad \times \prod_{j=1}^{6} H_j^{m_j}\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t+\tau\big)
\end{aligned}
$$

and

$$
\begin{aligned}
(84)\qquad & \frac{\mathrm{d}H_j\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t+\tau\big)}{\mathrm{d}t} = \\
&\quad = \lambda_j\big[G\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t+\tau\big) - \\
&\qquad - H_j\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t+\tau\big)\big].
\end{aligned}
$$

In (83), the Δ functions are defined similarly to (37) and (38), namely

$$\Delta(t, \mathrm{d}t) = \begin{cases} 1, & \text{for } 0 \le t \le \mathrm{d}t, \\ 0, & \text{otherwise}, \end{cases} \tag{85}$$

and

$$\bar{\Delta}_{12} = 1 - \Delta(t, \mathrm{d}t) - \Delta(t+\tau, \mathrm{d}t). \tag{86}$$

Two remarks are in order regarding eqs. (83)–(86). The first is that in (83), the total derivative with respect to t appears on the r.h.s. which thus affects two arguments. This is the consequence of the fact that we have transferred the derivative from the initial time to the terminal times by choosing $t_0 = 0$. The second is that it is assumed that $\Delta(t, \mathrm{d}t)$ and $\Delta(t+\tau, \mathrm{d}t)$ are non-overlapping, *i.e.* $\tau > \mathrm{d}t$. This only affects the calculation of the correlation in detector counts. In experiments the condition $\tau > \mathrm{d}t$ is always fulfilled since one only considers joint count statistics in different time bins but not in the same one.

Again, the generating function of the distribution induced by one single precursor from the j-th group, H_j, can be expressed explicitly in terms of G by solving (84). As mentioned, only the case $t > 0$ needs to be considered. The result, satisfying the initial condition (82), is

$$\begin{aligned} &H_j\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t+\tau\big) = \\ &\quad = \lambda_j \int_0^t e^{-\lambda_j(t-t')} \times \\ &\qquad \times G\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t'; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t'+\tau\big)\, \mathrm{d}t' + \\ &\quad + y_{1j} e^{-\lambda_j t} H_j\big(x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, \tau\big). \end{aligned} \tag{87}$$

Using the one-point solution (43) of H_j for the last term of the above, one obtains

$$\begin{aligned} &H_j\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t+\tau\big) = \\ &\quad = \lambda_j \int_0^t e^{-\lambda_j(t-t')} \times \\ &\qquad \times G\big(x_1, y_{11}, y_{12}, \dots, y_{16}, z_1, t'; x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t'+\tau\big)\, \mathrm{d}t' + \\ &\quad + y_{1j} e^{-\lambda_j t} \left\{ \lambda_j \int_0^\tau e^{-\lambda_j(\tau-t')} G\big(x_2, y_{21}, y_{22}, \dots, y_{26}, z_2, t'\big)\, \mathrm{d}t' + y_{2j} e^{-\lambda_j \tau} \right\}. \end{aligned} \tag{88}$$

In the present case, it is obviously not practical to substitute the above solution back into (83). Rather, it is more convenient to keep it separately.

The first moment of the neutron number, *i.e.*

$$(89)\qquad N(t) \equiv \langle N(t) \rangle = \\ = \frac{\partial G\big(x_1, y_{11}, y_{12}, \ldots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \ldots, y_{26}, z_2, t+\tau\big)}{\partial x_1}\bigg|_{x_i, y_{ij}, z_i = 1},$$

will be exactly the same as before, *i.e.* eq. (46) with the initial condition added in form of a δ-function. Hence the solution for $N(t)$, eq. (52), will also be the same. Calculating the first moment of the detector counts, *i.e.*

$$(90)\qquad Z(t) \equiv \langle Z(t) \rangle = \\ = \frac{\partial G\big(x_1, y_{11}, y_{12}, \ldots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \ldots, y_{26}, z_2, t+\tau\big)}{\partial z_1}\bigg|_{x_i, y_{ij}, z_i = 1},$$

will also yield a solution similar to the previous case:

$$(91)\qquad Z(t) = \lambda_\mathrm{d} \int_0^t N(t-t') \Delta(t', \mathrm{d}t)\, \mathrm{d}t'.$$

Due to the infinitesimal value of dt, (91) can be written as

$$(92)\qquad Z(t) = \varepsilon \lambda_\mathrm{f}\, \mathrm{d}t N(t)$$

and the stationary first moment of the detector counts in a system with a source is given as

$$(93)\qquad \tilde{Z} = \varepsilon \lambda_\mathrm{f}\, \mathrm{d}t \frac{\bar{q} S \Lambda}{-\rho}.$$

We turn now to the second moments, *i.e.* the correlations. For the neutron number we define

$$(94)\qquad M_{NN}(t,\tau) \equiv \langle N(t) N(t+\tau) \rangle = \\ = \frac{\partial^2 G\big(x_1, y_{11}, y_{12}, \ldots, y_{16}, z_1, t; x_2, y_{21}, y_{22}, \ldots, y_{26}, z_2, t+\tau\big)}{\partial x_1 \partial x_2}\bigg|_{x_i, y_{ij}, z_i = 1}.$$

Then, one obtains for M_{NN} the equation

$$(95)\qquad \frac{\mathrm{d}M_{NN}(t,\tau)}{\mathrm{d}t} = \lambda_\mathrm{f} \nu (1-\beta) M_{NN}(t,\tau) + \\ + \lambda_\mathrm{f} \sum_{j=1}^{6} \nu \beta_j \lambda_j \int_0^t e^{-\lambda_j (t-t')} M_{NN}(t',\tau)\, \mathrm{d}t' - \\ - \lambda_\mathrm{a} M_{NN}(t,\tau) + Q_{NN}(t,\tau) + \delta(t) N(\tau),$$

where $Q_{NN}(t,\tau)$ is given as

$$
\begin{aligned}
(96)\quad Q_{NN}(t,\tau) &= \\
&= \lambda_{\mathrm{f}}\Bigg[\big\langle \nu_p(\nu_p-1)\big\rangle N(t)N(t+\tau) + \\
&+ \sum_{j=1}^{6}\big\langle \nu_p\nu_{d_j}\big\rangle \Bigg\{ N(t)\lambda_j\int_0^{t+\tau} e^{-\lambda_j(t+\tau-t'')}N(t'')\,\mathrm{d}t'' + \\
&\qquad + N(t+\tau)\lambda_j\int_0^{t} e^{-\lambda_j(t-t')}N(t')\,\mathrm{d}t'\Bigg\} + \\
&+ \sum_{j=1}^{6}\big\langle \nu_{d_j}\big(\nu_{d_j}-1\big)\big\rangle {\lambda_j}^2\int_0^{t} e^{-\lambda_j(t-t')}N(t')\,\mathrm{d}t'\int_0^{t+\tau} e^{-\lambda_j(t+\tau-t'')}N(t'')\,\mathrm{d}t'' + \\
&+ \sum_{\substack{i,j=1\\ i<j}}^{6}\big\langle \nu_{d_i}\nu_{d_j}\big\rangle \lambda_i\lambda_j\Bigg\{\int_0^{t} e^{-\lambda_i(t-t')}N(t')\,\mathrm{d}t'\int_0^{t+\tau} e^{-\lambda_j(t+\tau-t'')}N(t'')\,\mathrm{d}t'' + \\
&\qquad + \int_0^{t} e^{-\lambda_j(t-t')}N(t')\,\mathrm{d}t'\int_0^{t+\tau} e^{-\lambda_i(t+\tau-t'')}N(t'')\,\mathrm{d}t''\Bigg\}\Bigg].
\end{aligned}
$$

In (95), the initial condition

$$(97)\qquad M_{NN}(0,\tau) = N(\tau),$$

which can be obtained from (80) and (94) (or by simple heuristic reasoning), was again added in form of a δ-function.

Here it is seen again that eq. (95) for $M_{NN}(t,\tau)$ is the same as eq. (46) for $N(t)$, except for the source term, and that $N(t)$ is the Green's function for $M_{NN}(t,\tau)$. Since the inhomogeneous term of (95) is now equal to $Q_{NN}(t,\tau)+\delta(t)N(\tau)$, one will have

$$(98)\qquad M_{NN}(t,\tau) = \int_0^t \big[Q_{NN}(t',\tau)+\delta(t')N(\tau)\big]N(t-t')\,\mathrm{d}t'$$

and using this in (25) will result in

$$(99)\qquad \tilde{C}_{NN}(\tau) = \tilde{N}\int_0^\infty Q_{NN}(t,\tau)\,\mathrm{d}t + \tilde{N}N(\tau) + \langle q(q-1)\rangle S\int_0^\infty N(t)N(t+\tau)\,\mathrm{d}t.$$

The concrete value of this expression is of no interest here, thus the integrals in (99) will not be evaluated. We just note that by $N(0)=1$ and utilizing the concrete form of $Q_{NN}(t,\tau)$, one can show that for $\tau=0$ the above reproduces the stationary value of the variance of the neutron number in a system with a source, *i.e.*

$$(100)\qquad \tilde{C}_{NN}(0) = \tilde{\sigma}^2_{NN} = \tilde{\mu}_{NN} + \tilde{N}.$$

Our concern of interest is to calculate the correlations in the detector counts, *i.e.* the Rossi-alpha formula. Thus, similarly to (94), we write

$$M_{ZZ}(t,\tau) \equiv \langle Z(t)Z(t+\tau)\rangle = \\ = \frac{\partial^2 G\big(x_1,y_{11},y_{12},\ldots,y_{16},z_1,t;x_2,y_{21},y_{22},\ldots,y_{26},z_2,t+\tau\big)}{\partial z_1 \partial z_2}\bigg|_{x_i,y_{ij},z_i=1}. \tag{101}$$

For this quantity, a result similar to (95) is obtained, *viz.*

$$\frac{\mathrm{d}M_{ZZ}(t,\tau)}{\mathrm{d}t} = \\ = \lambda_\mathrm{f}\nu(1-\beta)M_{ZZ}(t,\tau) + \\ + \lambda_\mathrm{f}\sum_{j=1}^{6}\nu\beta_j\lambda_j\int_0^t e^{-\lambda_j(t-t')}M_{NN}(t',\tau)\,\mathrm{d}t' - \lambda_\mathrm{a}M_{ZZ}(t,\tau) + Q_{ZZ}(t,\tau), \tag{102}$$

where

$$Q_{ZZ}(t,\tau) = \\ = \lambda_\mathrm{f}\Bigg[\big\langle\nu_p(\nu_p-1)\big\rangle Z(t)Z(t+\tau) + \\ + \sum_{j=1}^{6}\big\langle\nu_p\nu_{d_j}\big\rangle\bigg\{Z(t)\lambda_j\int_0^{t+\tau}e^{-\lambda_j(t+\tau-t'')}Z(t'')\,\mathrm{d}t'' + \\ + Z(t+\tau)\lambda_j\int_0^t e^{-\lambda_j(t-t')}Z(t')\,\mathrm{d}t'\bigg\} + \\ + \sum_{j=1}^{6}\big\langle\nu_{d_j}\big(\nu_{d_j}-1\big)\big\rangle{\lambda_j}^2\int_0^t e^{-\lambda_j(t-t')}Z(t')\,\mathrm{d}t'\int_0^{t+\tau}e^{-\lambda_j(t+\tau-t'')}Z(t'')\,\mathrm{d}t'' + \\ + \sum_{\substack{i,j=1\\ i<j}}^{6}\big\langle\nu_{d_i}\nu_{d_j}\big\rangle\lambda_i\lambda_j\bigg\{\int_0^t e^{-\lambda_i(t-t')}Z(t')\,\mathrm{d}t'\int_0^{t+\tau}e^{-\lambda_j(t+\tau-t'')}Z(t'')\,\mathrm{d}t'' + \\ + \int_0^t e^{-\lambda_j(t-t')}Z(t')\,\mathrm{d}t'\int_0^{t+\tau}e^{-\lambda_i(t+\tau-t'')}Z(t'')\,\mathrm{d}t''\bigg\}\Bigg]. \tag{103}$$

Since from (80) one obtains

$$M_{ZZ}(0,\tau) = 0, \tag{104}$$

there will be no δ-function on the r.h.s. of the equation for M_{ZZ}. Thus M_{ZZ} is given as a convolution between $N(t)$ and $Q_{ZZ}(t,\tau)$, and for the correlated counts in a stationary system with a source, one will have

$$\tilde{C}_{ZZ}(\tau) = \tilde{N}\int_0^\infty Q_{ZZ}(t,\tau)\,\mathrm{d}t + \langle q(q-1)\rangle S\int_0^\infty Z(t)Z(t+\tau)\,\mathrm{d}t. \tag{105}$$

This equation has one term less on the r.h.s. than eq. (99), which expresses the fact that the detected initial neutron will have no progenies and thus cannot create correlations, whereas without detection the initial neutron will start a chain.

By virtue of eq. (92), *i.e.* the fact that we use infinitesimal time bins, from (103) one will have the simple relation

$$Q_{ZZ}(t,\tau) = \varepsilon^2 \lambda_{\mathrm{f}}^2 (\mathrm{d}t)^2 Q_{NN}(t,\tau) \tag{106}$$

and finally

$$\tilde{C}_{ZZ}(\tau) = \varepsilon^2 \lambda_{\mathrm{f}}^2 (\mathrm{d}t)^2 \left(\tilde{N} \int_0^\infty Q_{NN}(t,\tau)\,\mathrm{d}t + \langle q(q-1)\rangle S \int_0^\infty N(t)N(t+\tau)\,\mathrm{d}t \right). \tag{107}$$

In view of the relationship between the cross terms, *i.e.*

$$\begin{aligned} \int_0^\infty \Big\{ N(t) \int_0^{t+\tau} e^{-\lambda_j (t+\tau-t'')} N(t'')\,\mathrm{d}t'' + \\ + N(t+\tau) \int_0^t e^{-\lambda_j (t-t')} N(t')\,\mathrm{d}t' \Big\} \mathrm{d}t = \Psi_j(\tau), \end{aligned} \tag{108}$$

$$\int_0^\infty \left\{ \lambda_j \int_0^t e^{-\lambda_j (t-t')} N(t')\,\mathrm{d}t' \int_0^{t+\tau} e^{-\lambda_j (t+\tau-t'')} N(t'')\,\mathrm{d}t'' \right\} \mathrm{d}t = \frac{1}{2}\Psi_j(\tau), \tag{109}$$

and

$$\begin{aligned} \int_0^\infty \left\{ \int_0^t e^{-\lambda_i (t-t')} N(t')\,\mathrm{d}t' \int_0^{t+\tau} e^{-\lambda_j (t+\tau-t'')} N(t'')\,\mathrm{d}t'' \right\} \mathrm{d}t + \\ + \int_0^\infty \left\{ \int_0^t e^{-\lambda_j (t-t')} N(t')\,\mathrm{d}t' \int_0^{t+\tau} e^{-\lambda_i (t+\tau-t'')} N(t'')\,\mathrm{d}t'' \right\} \mathrm{d}t = \\ = \frac{1}{\lambda_i + \lambda_j} \big[\Psi_i(\tau) + \Psi_j(\tau) \big], \end{aligned} \tag{110}$$

this formula can be compactly written as

$$\begin{aligned} \tilde{C}_{ZZ}(\tau) = \varepsilon^2 \lambda_{\mathrm{f}}^2 (\mathrm{d}t)^2 \Bigg[\Big\{ \tilde{N} \lambda_{\mathrm{f}} \big\langle \nu_p (\nu_p - 1) \big\rangle + \langle q(q-1)\rangle S \Big\} \int_0^\infty N(t)N(t+\tau)\,\mathrm{d}t + \\ + \sum_{j=1}^{6} \lambda_j \left\{ \big\langle \nu_p \nu_{d_j} \big\rangle + \frac{1}{2} \big(\nu_{d_j} \big(\nu_{d_j} - 1 \big) \big) \right\} \Psi_j(\tau) + \\ + \sum_{\substack{i,j=1 \\ i<j}}^{6} \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j} \big\langle \nu_{d_i} \nu_{d_j} \big\rangle \big\{ \Psi_i(\tau) + \Psi_j(\tau) \big\} \Bigg]. \end{aligned} \tag{111}$$

Here

$$\Psi_j(\tau) = \int_0^\infty e^{-\lambda_j(t+\tau)} N(t) \left(\int_0^{(t+\tau)} e^{\lambda_j t'} N(t') \,\mathrm{d}t' \right) \mathrm{d}t + \\ + \int_{-\tau}^\infty e^{-\lambda_j(t+\tau)} N(t+2\tau) \left(\int_0^{(t+\tau)} e^{\lambda_j t'} N(t') \,\mathrm{d}t' \right) \mathrm{d}t. \tag{112}$$

The Rossi formula is usually written in the form

$$P_{\mathrm{rossi}}(t)\,\mathrm{d}t = \frac{\tilde{C}_{ZZ}(t)}{\tilde{Z}} \tag{113}$$

Using (111), it is only necessary to calculate $\int_0^\infty N(t)N(t+\tau)\,\mathrm{d}t$ and $\Psi_j(\tau)$. They were evaluated with a Mathematica notebook analytically.

The result by re-denoting τ as t can be compactly written as follows:

$$P_{\mathrm{rossi}}(t)\,\mathrm{d}t = \\ = \varepsilon\lambda_{\mathrm{f}}^2\,\mathrm{d}t \Bigg\{ \left[\big\langle \nu_p\big(\nu_p - 1\big)\big\rangle + \langle q(q-1)\rangle \frac{\nu}{\bar{q}}(-\rho) \right] \sum_{i=0}^{6} \omega_i f_i(t) - \\ - \sum_{j=1}^{6} \lambda_j^2 \Big(2\langle \nu_p \nu_{d_j}\rangle + \big\langle \nu_{d_j}\big(\nu_{d_j} - 1\big)\big\rangle \Big) \sum_{i=0}^{6} \frac{\omega_i f_i(t)}{s_i^2 - \lambda_j^2} - \\ - 2 \sum_{\substack{i,j=1 \\ i<j}}^{6} \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j} \langle \nu_{d_i} \nu_{d_j}\rangle \left(\lambda_i \sum_{k=0}^{6} \frac{\omega_k f_k(t)}{s_k^2 - \lambda_i^2} + \lambda_j \sum_{k=0}^{6} \frac{\omega_k f_k(t)}{s_k^2 - \lambda_j^2} \right) \Bigg\}, \tag{114}$$

where the functions ω_i are exactly the same as in (71), whereas the functions $f_i(t)$ are defined as

$$f_i(t) \equiv -s_i e^{s_i t} = \alpha_i e^{-\alpha_i t}, \quad i = 0, 1, \ldots, 6. \tag{115}$$

This is the rigorous solution for the Rossi-alpha formula with multiple emission sources and by taking the delayed neutrons as well as their correlations into account.

Again, it can be shown that by condensing the six delayed neutron groups into one, the present formula is identical with the formula derived recently in [21].

It is also seen that the presence of the multiple source in the Rossi-alpha formula has the same consequences as in case of the Feynman-alpha formula. Namely, the time dependence of the formula is unchanged, only the amplitude is increased. The contribution of the source correlations (multiplicity) to the correlated detector counts has the same relative weight as in the case of the Feynman-alpha. Again, this property can be seen already in the starting equations, cf. eqs. (103) and (105), without deriving the final formula.

5. – Conclusions

The present results constitute the most detailed and rigorous calculations of the Feynman-alpha and Rossi-alpha formulae for systems with multiple emission sources (ADS), six delayed neutron groups and prompt-delayed and delayed-delayed correlations included. Besides the value of the results, it was demonstrated how the problem can be solved effectively with a combination of analytical techniques and symbolic algebra calculations, using Mathematica.

The results show that the dependence of the formulas on the counting time (Feynman-alpha) or on the time delay of the covariance (Rossi-alpha) is the same for a spallation source as it was in the case of traditional sources. It is only the amplitude of the corresponding expressions that is increased. The physical reason is that the time dependence of the formulas, *i.e.* the saturation time of the variance-to-mean and the decay of the covariance, is related to the dying-out of the individual chains induced by the source neutrons. This dying-out is only related to the properties of the system, but not the source. The amplitude of the variance-to-mean and the covariance, respectively, is determined by the production rate of correlated neutrons. In a traditional system this production takes place in the subcritical fission chains, whereas in case of a spallation source, an extra term exists since original correlations are generated already in the source. This is why the amplitude is increased. This increase of the amplitude is useful in practical applications, thus use of the Feynman- and Rossi-alpha methods is promising in future accelerator-based systems.

From the point of view of practical applications, the reactor physics model used is rather restrictive, *i.e.* an infinite, homogeneous and energy-independent model. For practical applications several of these limitations should be eliminated. In particular, the energy dependence may play a larger role than in traditional systems, due to the high energy of the source neutrons. Calculations in more complicated systems, however, lend little hope of analytical solutions and will require the use of Monte Carlo methods.

Appendix

Nomenclature

v = neutron velocity (constant).

$\bar{\nu}$ = average number of total neutrons per fission.

Σ_{c} = macroscopic cross-section of capture.

Σ_{f} = macroscopic cross-section of fission.

Σ_{d} = macroscopic cross-section of detection.

$\lambda_{\mathrm{c}} \equiv v \cdot \Sigma_{\mathrm{c}}$ = probability of capture per unit time per neutron.

$\lambda_{\mathrm{f}} \equiv v \cdot \Sigma_{\mathrm{f}}$ = probability of fission per unit time per neutron.

$\lambda_{\mathrm{d}} \equiv v \cdot \Sigma_{\mathrm{d}}$ = probability of detection per neutron and unit time.

$\lambda_{\mathrm{a}} = \lambda_{\mathrm{f}} + \lambda_{\mathrm{c}} + \lambda_{\mathrm{d}}$ is the total probability of neutron absorption per unit time per neutron.

λ_j = the j-th group delayed-neutron time constant.

$p(n, m_1, \ldots, m_6)$ = probability of emission of n prompt neutrons and m_j delayed neutron precursors from the j-th group per fission.

$p_q(n) =$ probability that n neutrons are generated in a source emission event.

$S =$ probability of an external-source neutron emission per unit time.

$\rho \equiv \frac{\nu\lambda_\mathrm{f}-\lambda_\mathrm{a}}{\nu\lambda_\mathrm{f}} = \frac{\nu\Sigma_\mathrm{f}-\Sigma_\mathrm{a}}{\nu\Sigma_\mathrm{f}}$ reactivity of the system.

$\Lambda = \frac{1}{\nu\lambda_\mathrm{f}}$.

$\varepsilon = \frac{\lambda_\mathrm{d}}{\lambda_\mathrm{f}}$ detector efficiency.

$\langle\nu_p(\nu_p-1)\rangle = \sum_n\sum_m n(n-1)p(n,m_1,\ldots,m_6)$.

$\langle\nu_p\nu_{d_j}\rangle = \sum_n\sum_{m_1}\cdots\sum_{m_6} nm_jp(n,m_1,\ldots,m_6)$.

$\langle\nu_{d_j}(\nu_{d_j}-1)\rangle \sum_n\sum_{m_1}\cdots\sum_{m_6} m_j(m_j-1)p(n,m_1,\ldots,m_6)$.

$\langle\nu_{d_i}\nu_{d_j}\rangle = \sum_n\sum_{m_1}\cdots\sum_{m_6} m_im_jp(n,m_1,\ldots,m_6)$.

$D_\nu = \frac{\langle\nu_p(\nu_p-1)\rangle}{\bar{\nu}_p^2}$ fission Diven factor.

$D_q = \frac{\langle q(q-1)\rangle}{\bar{q}^2}$ source Diven factor.

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