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On the Neutron Flux Flattening Problem

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ABSTRACT

In this paper, we present the critical neutron flux flattening problem governed by the critical transport equation in a nonuniform slab with periodic boundary conditions. Existence and uniqueness theorem of the optimal solution is shown in continuous function space.

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1. INTRODUCTION

Since 1970s, optimal control theory of distributed parameter systems has come into wide use in the nuclear field, both to nuclear design problems and to nuclear plant operation problems[1, 2]. One type of the problems is to minimize the deviation of the flux distribution from its average value by manipulating the fissile material distribution, which is the so-called flux flattening problem[1].

Consider the critical neutron distribution in a nonuniform slab of thickness 2a with periodic boundary conditions. The angular neutron flux flattening problem is to[cf. 3-6]

$$\text{minimize } I(\psi) = \int_{-a}^{a}\int_{-1}^{+1} |\psi(x,\mu) - M\psi|^2 dx d\mu, \tag{1}$$

while the state variable ψ is subject to the requirement for the conservation of the total output

$$\int_{-a}^{a}\int_{-1}^{+1} \sigma(x)u(x)\psi(x,\mu)dxd\mu = P = constant > 0, \tag{2}$$

and the state equations

$$\mu\frac{\partial}{\partial x}\psi(x,\mu) + \sigma(x)\psi(x,\mu) = \frac{\sigma(x)u(x)}{2}\int_{-1}^{+1}\psi(x,\mu')d\mu' \tag{3}$$

$$\psi(-a,\mu) = \psi(a,\mu), \quad |\mu| \le 1. \tag{4}$$

$$\psi(x,\mu) \ge 0, \ |x| \le a, \ |\mu| \le 1. \tag{5}$$

Where $M\psi$ denotes the mean value of ψ, that is $M\psi = \frac{1}{4a}\int_{-a}^{a}\int_{-1}^{+1}\psi(x,\mu)dxd\mu$. $\sigma(x)$ is the total cross section, and u(x) is the mean number of secondaries per collision.

We take u(x) as the controllable function, and it belongs to the admissible control set denoted by

$$U = \{u \in X : 0 < C_1 \le u(x) \le C_2\},$$

where C_1 and C_2 are positive constants, and X $\equiv L^\infty[-a,a]$, the usual Banach space consisting of all essentially bounded functions on [-a, a].

Let Y be the state space, and V denote the set

$V \equiv \{\psi \in Y$: there exists $u \in U$ such that (u, ψ) satisfies (2)-(5) $\}$.

Then the optimal control problem can be written as the minimization problem

$$I(\psi^*) = \min_{\psi\in V} I(\psi), \qquad \psi^* \in V, \tag{6}$$

If the Hilbert space $L^2([-a,a]\times[-1,1])$ is chosen as the state space, the similar problem has been discussed in [5]. And the existence-uniqueness theorem and optimality conditions for the optimal solution have been obtained. These results extended those given by Terney[7] in a symmetric, 1-D slab reactor described by one group theory. In order to disclose the difficulty that we are going to meet, we'd like to say a few words about the proof of the existence-uniqueness theorem in [5].

The authors showed that V is a closed convex set, $I(\cdot)$ is a strictly convex continuous functional in V, and $\lim_{\|\psi\|\to\infty} I(\psi) = \infty$. Therefore the existence of the optimal solution comes from the reflexiveness of $L^2([-a,a]\times[-1,1])$. And the uniqueness comes from the strict convexity of the performance index.

However, in order to consider the numerical analysis and computation, it is necessary to study the problem in continuous function space. The non-reflexiveness of the continuous function space make the existence of the optimal solution so difficult that we have to leave it as an open problem up to now.

In this paper, we are concerned with the neutron flux flattening problem.

What we mean by this is that the following performance index $F(\cdot)$, instead of $I(\cdot)$, is considered.

$$F(\psi) = \int_{-a}^{a} |\phi(x) - m\phi|^2 dx, \quad (7)$$

where $\phi = \frac{1}{2}\int_{-1}^{1} \psi(x,\mu)d\mu$ is the neutron flux, and $m\phi = \frac{1}{2a}\int_{-a}^{a} \phi(x)dx$.

The physical meaning of (7) is obvious. Actually it is expected that $\phi(x) = constant$. In this case, there is no leakage of neutron and thus the reactor has the least amount of nuclear fuel. So we say that the flatter the neutron flux is, the safer the reactor is.

For the new performance index $F(\cdot)$, the existence of the optimal solution becomes easier(see Theorem 3.1). But since $\int_{-1}^{1} \psi(x,\mu)d\mu = \int_{-1}^{1} \psi'(x,\mu)d\mu$ may be possible for different ψ and ψ', the strict convexity of $F(\cdot)$ vanishes so that the uniqueness of the optimal solution becomes more difficult.

In this paper, we show the existence-uniqueness theorem of the optimal solution with different methods from [5].

Before ending this section, we make the following assumption for the next sections.

(H) σ(x) is nonnegative continuous function on [-a, a], and $0 < \sigma_0 = \min_{x\in[-a,a]} \sigma(x) \leq \sigma(x) \leq \max_{x\in[-a,a]} \sigma(x) = \sigma_M$.

2. FORMULATION OF THE PROBLEM

Let Q = [-a, a], D = [-1, 0)$\cup$ (0, 1], G = $Q \times D$, and Y be the Banach space composed of all real continuous functions defined on G with the supremum norm $\|\cdot\| = \sup_{g\in G} |\cdot(g)|$.

For any $\psi \in V$, let

$$\phi = \tfrac{1}{2}\int_{-1}^{+1} \psi(x,\mu')d\mu'. \tag{8}$$

Then $\phi \in Z \equiv C(Q)$, the continuous function space with the maximum norm, and

$$\phi \geq 0, \tag{9}$$

$$\int_{-a}^{a} \sigma(x)u(x)\phi(x)dx = \tfrac{P}{2}, \tag{10}$$

and (3) and (4) lead to that

$$\text{(11)} \qquad \psi = \begin{cases} \{1 - exp[-\frac{1}{\mu}\int_{-a}^{a}\sigma(s)ds]\}^{-1}(J_1\sigma u\phi + J_2\sigma u\phi), & if\,\mu > 0, \\ u\phi, & if\,\mu = 0, \\ \{1 - exp[\frac{1}{\mu}\int_{-a}^{a}\sigma(s)ds]\}^{-1}(J_3\sigma u\phi + J_4\sigma u\phi), & if\,\mu < 0. \end{cases}$$

where

$$J_1\sigma u\phi(x,\mu) = \frac{1}{\mu}\int_{-a}^{x} exp[-\frac{1}{\mu}\int_{x'}^{x}\sigma(s)ds]\sigma(x')u(x')\phi(x')dx',$$

$$J_2\sigma u\phi(x,\mu) = \frac{1}{\mu}\int_{x}^{a} exp\{-\frac{1}{\mu}[\int_{x'}^{x}\sigma(s)ds + \int_{-a}^{a}\sigma(s)ds]\}\sigma(x')u(x')\phi(x')dx',$$

$$J_3\sigma u\phi(x,\mu) = -\frac{1}{\mu}\int_{x}^{a} exp[\frac{1}{\mu}\int_{x}^{x'}\sigma(s)ds]\sigma(x')u(x')\phi(x')dx',$$

$$J_4\sigma u\phi(x,\mu) = -\frac{1}{\mu}\int_{-a}^{x} exp\{\frac{1}{\mu}[\int_{x}^{x'}\sigma(s)ds + \int_{-a}^{a}\sigma(s)ds]\}\sigma(x')u(x')\phi(x')dx',$$

Integrating (11) with respect to μ, and replacing μ with $\frac{1}{t}$, we get

$$\phi = \int_{-a}^{a} K(x,x')u(x')\phi(x')dx', \tag{12}$$

where

$$K(x,x') = \tfrac{1}{2}\sigma(x')\int_{1}^{+\infty}\tfrac{dt}{t}\{1 - \exp[-t\int_{-a}^{a}\sigma(s)ds]\}^{-1}\times$$

$$\{\exp(-|\int_x^{x'}\sigma(s)ds|t)+\exp(-[\int_{-a}^{a}\sigma(s)ds-|\int_x^{x'}\sigma(s)ds|]t)\}$$

Let W denote the set $\{\phi\in Z$: there exists $u\in U$ such that (u, ϕ) satisfies (9), (10) and (12)$\}$. Then the corresponding optimal problem to the index (7) can be written as the minimization problem

$$\tilde{F}(\phi^*)=\min_{\phi\in W}\tilde{F}(\phi), \tag{13}$$

3. EXISTENCE-UNIQUENESS THEOREM

Let us define integral operators K_1 from Z to Z, K_2 from X to Z, and K_3 from $L^2(Q)$ to $L^2(Q)$, the usual Hilbert space composed of square integrable functions, by

$$K_i\bullet=\int_{-a}^{a}K(x,x')\bullet(x')dx',\quad i=1,2,3 \tag{14}$$

Lemma 3.1 (1) if $\phi=K_i\phi$ and $\phi\geq 0$, then $\phi>0, i=1,2,3$;

(2) K_1 is a compact operator from Z to Z;

(3) K_2 is a compact operator from X to Z.

(4) K_3 is a compact operator from $L^2(Q)$ to $L^2(Q)$.

Proof. (1) is obvious.

(2) Considering the transform

$$\mathrm{y}=\mathrm{f(s)}=\int_{-a}^{x}\sigma(s)ds-\frac{\Delta_0}{2},\ \Delta_0=\int_{-a}^{a}\sigma(s)ds, \tag{15}$$

we have

$$\int_{-a}^{a}K(x,x')dx'$$

$$=\frac{1}{2}\int_1^{+\infty}\frac{dt}{t}\{1-\exp(-\Delta_0 t)\}^{-1}\int_{-\frac{\Delta_0}{2}}^{\frac{\Delta_0}{2}}\{\exp(-|y-y'|t)+\exp(-[\Delta_0-|y-y'|]t)\}dy'$$

$$= 1. \tag{16}$$

Hence, for every $\phi \in Z$,

$$|K\phi(x)| \le \int_{-a}^{a} K(x,x')|\phi(x')|dx' \le \|\phi\|. \tag{17}$$

This shows that for any bounded set S in Z, KS is uniformly bounded.

Further, for any $\phi \in S$, $x_1, x_2 \in Q$, by the same transform with (15), a tedious calculation shows that

$$|K\phi(x_1) - K\phi(x_2)|$$

$$\le \|\phi\| \int_1^{+\infty} \frac{dt}{t^2}[1 - \exp(-\Delta_0 t)]^{-1}\alpha(y_1, y_2, t).$$

where

$$\alpha(y_1,y_2,t) = |2 - 2\exp(-\tfrac{|y_1-y_2|}{2}t)| + |\exp[-(y_2 + \tfrac{\Delta_0}{2})t] - \exp[-(y_1 + \tfrac{\Delta_0}{2})t]| + |\exp[-(\tfrac{\Delta_0}{2} - y_2)t] - \exp[-(\tfrac{\Delta_0}{2} - y_1)t]| + |2\exp(-\Delta_0 t) - 2\exp[-(\Delta_0 - \tfrac{|y_2-y_1|}{2})t]|. \tag{18}$$

Since $\alpha(y_1, y_2, t) \le 6$, for any $\varepsilon > 0$, there exists $T > 0$, independent in ϕ, such that

$$\int_T^{+\infty} \frac{dt}{t^2}[1 - \exp(-\Delta_0 t)]^{-1}\alpha(y_1, y_2, t) < \frac{\varepsilon}{2} \tag{19}$$

Since $|\exp(x) - 1| \le |x|, x \le 0$, it is easy to check that $\alpha(y_1, y_2, t) \le 4|y_2 - y_1|t$.

Thus

$$\int_1^T \frac{dt}{t^2}[1 - \exp(-\Delta_0 t)]^{-1}\alpha(y_1, y_2, t) \le 4|y_2 - y_1| T \int_1^T \frac{dt}{t^2}[1 - \exp(-\Delta_0 t)]^{-1} \tag{20}$$

From (19) and (20), one may easily see that KS is an equi-continuous set. Therefore KS is a locally compact set in Z, and K is a compact operator from Z to Z;

(3) may be obtained by the same procedure with (2).

(4) has been shown in Ref.8.

Lemma 3.2[cf. 11] W $\neq \emptyset$ iff $C_2 \geq \frac{1}{\gamma(K_1)}$, where $\gamma(K_1)$ denotes the spectral radius of K_1.

Proof. The same procedure with Lemma 6 in Ref. 9 shows that $\gamma(K_1) > 0$. Theorem 2.1 in Ref. 10 shows that $\gamma(K_1)$ is an eigenvalue of K_1 with nonnegative eigenfunction ϕ. Let $\hat{\phi}(x) = \frac{P\gamma(K_1)\phi(x)}{2\int_{-a}^{a}\sigma(x)\phi(x)dx}$, then $\hat{\phi} \in W$ with the corresponding $\hat{u}(x) \equiv \frac{1}{\gamma(K_1)}$.

On the other hand, if W $\neq \emptyset$, there exists u $\in$ U and $\phi > 0$ [Lemma 3.1(1)] such that $\phi = K_1 u\phi \leq MK_1\phi$. Therefore the same reason with Lemma 6 in Ref. 9 shows that M$\geq \frac{1}{\gamma(K_1)}$.

Lemma 3.3 W is a closed convex set in Z.

Proof. It is easy to show that W is a convex set in Z [cf. 11].

Suppose that $\{\phi_n\} \subset$W, and ϕ_n strongly converges to $\phi_0 \in Z$. By the definition of W, there exists$\{u_n\} \subset$ U such that $\phi_n = K_3 u_n \phi_n$.

Since U is a weak* closed set in X = $(L^1(Q))^*$, the conjugate space of the usual absolutely integrable function space $L^1(Q)$, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u_0 \in U$ such that

$$\int_{-a}^{a} u_n(x)f(x)dx \to \int_{-a}^{a} u_0(x)f(x)dx, \text{ for every f} \in L^1(Q), \tag{21}$$

And it is easy to see that

$$\int_{-a}^{a} u_n(x)\phi_n(x)f(x)dx \to \int_{-a}^{a} u_0(x)\phi_0(x)f(x)dx, \text{ for every f} \in L^1(Q), \tag{22}$$

In particular,

$$\int_{-a}^{a} u_n(x)\phi_n(x)f(x)dx \to \int_{-a}^{a} u_0(x)\phi_0(x)f(x)dx, \text{ for every f} \in L^2(Q), \tag{23}$$

Lemma 3.1(4) concludes $\phi_0 = K_3 u_0 \phi_0$. That is

$$\phi_0(x) = \int_{-a}^{a} K(x,x')u_0(x')\phi_0(x')dx', \tag{24}$$

Therefore it is easy to find $\phi_0 \in W$.

Lemma 3.4 $\tilde{F}(\cdot)$ is a strictly convex functional in W.

Proof. For any ϕ_1, ϕ_2 in W, and $0 < t < 1$,

$$\tilde{F}(t\phi_1 + (1-t)\phi_2)$$

$$= \int_{-a}^{a}[t(\phi_1(x) - m\phi_1) + (1-t)(\phi_2(x) - m\phi_2)]^2 dx$$

$$= t\int_{-a}^{a}(\phi_1(x) - m\phi_1)^2 dx + (1-t)\int_{-a}^{a}(\phi_2(x) - m\phi_2)^2 dx - t(1-t)\int_{-a}^{a}[(\phi_1(x) - m\phi_1 - (\phi_2(x) - m\phi_2)]^2 dx, \tag{25}$$

If $\int_{-a}^{a}[(\phi_1(x) - m\phi_1 - (\phi_2(x) - m\phi_2)]^2 dx = 0$, $(\phi_1(x) - m\phi_1) - (\phi_2(x) - m\phi_2) = 0$. So $\phi_1(x) - \phi_2(x) = m\phi_1 - m\phi_2 = constant$, and the boundary conditions conclude that $\phi_1(x) = \phi_2(x)$. Therefore we have shown the lemma.

Theorem 3.1 If $C_2 \geq \frac{1}{\gamma(K_1)}$, where $\gamma(K_1)$ denotes the spectral radius of K_1, there is unique ϕ^* in W such that

$$\tilde{F}(\phi^*) = \min_{\phi \in W} \tilde{F}(\phi), \tag{26}$$

Proof. The uniqueness comes directly from the strict convexity of $\tilde{F}$. In the following, we shall show the existence of the optimal solution.

Let $\{\phi_n\}$ be a minimized sequence, that is to say that $\phi_n \in W$, and $\lim_{n\to\infty} \tilde{F}(\phi_n) = \inf_{\phi \in W} \tilde{F}(\phi)$. So

$$\int_{-a}^{a}(m\phi_n - \phi_n(x))^2 dx \leq C_3, \tag{27}$$

Since

$$m\phi_n \leq \frac{1}{\int_{-a}^{a}\sigma(x)u_n(x)dx}[\int_{-a}^{a}(m\phi_n - \phi_n(x))\sigma(x)u_n(x)dx + \int_{-a}^{a}\phi_n(x)\sigma(x)u_n(x)dx]$$

$$\leq \frac{1}{C_1 \int_{-a}^{a} \sigma(x)dx}[C_4(\int_{-a}^{a}(m\phi_n - \phi_n(x))^2 dx)^{1/2} + \frac{P}{2}]$$

$$\leq C_5, \tag{28}$$

$$\int_{-a}^{a} \phi_n^2 dx \leq 2[\int_{-a}^{a}(\phi_n(x) - m\phi_n)^2 dx + \int_{-a}^{a}(m\phi_n)^2 dx] \leq C_6. \tag{29}$$

And since

$$\int_{-a}^{a} K(x,x')^2 dx'$$

$$\leq \int_{-a}^{a} dx' \int_1^\infty \frac{dt}{t^{4/3}}[1 - \exp(-\Delta_0 t)]^{-2} \int_1^\infty \frac{dt}{t^{2/3}}\{\exp(-2|\int_x^{x'} \sigma(s)ds|t) + 2\exp(-\Delta_0 t)$$

$$+ \exp(-2[\int_{-a}^{a} \sigma(s)ds - |\int_x^{x'} \sigma(s)ds|]t)\}$$

$$\leq \int_1^\infty \frac{dt}{t^{4/3}}[1 - \exp(-\Delta_0 t)]^{-2}\{\int_1^\infty \frac{dt}{t^{5/3}}[1 - \exp(-2\Delta_0 t)] + 2\int_1^\infty \frac{dt}{t^{2/3}} \exp(-2\Delta_0 t)\}$$

$$\leq C_7 \tag{30}$$

$$|\phi_n(x)|^2 \leq \int_{-a}^{a}[K(x,x')u_n(x')]^2 \int_{-a}^{a} \phi_n^2(x')dx' \leq C_8. \tag{31}$$

Where C_i, i =1, $\cdots$, 8, are constants. So $\{u_n\phi_n\}$ is bounded in $L^\infty(Q)$.

Since $\phi_n = K_2 u_n \phi_n$, and K_2 is compact, there exists a convergent subsequence of $\{\phi_n\}$, still denoted by $\{\phi_n\}$. Suppose ϕ_n converges to ϕ_0 in Z. Since W is closed, $\phi_0 \in W$. Therefore

$$\tilde{F}(\phi_0) = \lim_{n\to\infty} \tilde{F}(\phi_n) = \inf_{\phi \in W} \tilde{F}(\phi), \tag{32}$$

Now let's turn to the original problem, we have

Theorem 3.2 If $C_2 \geq \frac{1}{\gamma(K_1)}$, where $\gamma(K_1)$ denotes the spectral radius of K_1, there is unique (u^*, ψ^*) such that

$$F(\psi^*) = \min_{\psi \in V} F(\psi) \tag{33}$$

while (u^*, ψ^*) satisfies the following equations:

$$\int_{-a}^{a} \int_{-1}^{+1} \sigma(x)u^*(x)\psi^*(x,\mu)dxd\mu = P = constant > 0, \tag{34}$$

$$\mu\frac{\partial}{\partial x}\psi^*(x,\mu)+\sigma(x)\psi^*(x,\mu)=\frac{\sigma(x)u^*(x)}{2}\int_{-1}^{+1}\psi^*(x,\mu')d\mu' \tag{35}$$

$$\psi^*(-a,\mu)=\psi^*(a,\mu),\ \ |\mu|\le 1. \tag{36}$$

$$\psi^*(x,\mu)\ge 0,\ |x|\le a,\ |\mu|\le 1. \tag{37}$$

Proof. Let ϕ^* be the unique optimal solution with corresponding $u^*\in U$. Then (u^*,ϕ^*) satisfies equations (9), (10) and (12).

Let

$$\psi^*=\begin{cases}\{1-exp[-\frac{1}{\mu}\int_{-a}^{a}\sigma(s)ds]\}^{-1}(J_1\sigma u^*\phi^*+J_2\sigma u^*\phi^*), & if\,\mu>0,\\ u^*\phi^*, & if\,\mu=0,\\ \{1-exp[\frac{1}{\mu}\int_{-a}^{a}\sigma(s)ds]\}^{-1}(J_3\sigma u^*\phi^*+J_4\sigma u^*\phi^*), & if\,\mu<0.\end{cases} \tag{38}$$

Then (u^*,ψ^*) satisfies equations (2)-(5), and

$$F(\psi^*)=\min_{\psi\in V}F(\psi) \tag{39}$$

Suppose there is another (u',ψ') satisfies equations (2)-(5) and (39).

Let

$$\psi_0(x,\mu)=\psi'(x,\mu)-\psi^*(x,\mu) \tag{40}$$

Then

$$\frac{1}{2}\int_{-1}^{1}\psi'(x,\mu)d\mu=\frac{1}{2}\int_{-1}^{1}\psi^*(x,\mu)d\mu=\phi^*(x) \tag{41}$$

$$\mu\frac{\partial}{\partial x}\psi_0(x,\mu)+\sigma(x)\psi_0(x,\mu)=\sigma(x)[u'(x)-u^*(x)]\phi^*(x) \tag{42}$$

$$\psi_0(-a,\mu)=\psi_0(a,\mu),\ \ |\mu|\le 1. \tag{43}$$

Considering the transform (15) and denoting $\psi_0(f^{-1}(s),\mu)$ and $\frac{1}{\sigma(f^{-1}(s))}[u'(f^{-1}(s))-u^*(f^{-1}(s))]\phi^*(f^{-1}(s))$ by $\tilde{\psi}(s,\mu)$ and $\tilde{\phi}(s)$, (42) becomes

$$\mu\frac{\partial}{\partial s}\tilde{\psi}(s,\mu)+\tilde{\psi}(s,\mu)=\tilde{\phi}(s) \tag{44}$$

$$\tilde{\psi}(-\tfrac{\Delta_0}{2},\mu)=\tilde{\psi}(-\tfrac{\Delta_0}{2},\mu),\ \ |\mu|\le 1. \tag{45}$$

$$\int_{-1}^{1} \tilde{\psi}(s,\mu)d\mu = 0, \tag{46}$$

Let $e_n(s) = \frac{1}{\sqrt{\Delta_0}}\exp(i\frac{2ns\pi}{\Delta_0}), (n = 0, \pm1, \pm2, \cdots)$. Then $\{e_n\}$ is a complete orthonormal base in $L^2[-\frac{\Delta_0}{2}, \frac{\Delta_0}{2}]$.

$\mu\frac{\partial}{\partial s}\tilde{\psi}(s,\mu)$ and $\tilde{\psi}(s,,\mu)$ can be expressed by

$$\tilde{\psi}(s,\mu) = \sum_{n=-\infty}^{n=\infty} e_n(s) f_n(\mu) \tag{47}$$

$$\mu\frac{\partial}{\partial s}\tilde{\psi}(s,\mu) = \sum_{n=-\infty}^{n=\infty} e_n(s) g_n(\mu) \tag{48}$$

where

$$f_n(\mu) = \int_{-\frac{\Delta_0}{2}}^{\frac{\Delta_0}{2}} e_n(s)\tilde{\psi}(s,\mu)ds, \tag{49}$$

$$g_n(\mu) = \int_{-\frac{\Delta_0}{2}}^{\frac{\Delta_0}{2}} e_n(s)\mu\frac{\partial}{\partial s}\tilde{\psi}(s,\mu)ds, \tag{50}$$

The integration of (50) by parts causes that

$$g_n(\mu) = -\frac{2n\pi i}{\Delta_0}\mu f_n(\mu) \tag{51}$$

Suppose

$$\tilde{\phi}(s) = \sum_{n=-\infty}^{n=\infty} a_n e_n(s) \tag{52}$$

Then (44) becomes that

$$\sum_{n=-\infty}^{n=\infty}(1 - \frac{2n\pi i}{\Delta_0}\mu)e_n(s)f_n(\mu) = \sum_{n=-\infty}^{n=\infty} a_n e_n(s) \tag{53}$$

So

$$(1 - \frac{2n\pi i}{\Delta_0}\mu)f_n(\mu) = a_n, \tag{54}$$

$$\int_{-1}^{1} f_n(\mu)d\mu = \frac{\Delta_0 a_n i}{2n\pi} ln\frac{\Delta_0 - 2n\pi i}{\Delta_0 + 2n\pi i}, n = \pm1, \pm2, \cdots \tag{55}$$

$$\int_{-1}^{1} f_0(\mu)d\mu = 2a_0 \tag{56}$$

On the other hand, (46) and (47) lead to $\int_{-1}^{1} f_n(\mu)d\mu = 0, n = 0, \pm1, \pm2, \cdots$. Therefore $a_n = 0, n = 0, \pm1, \pm2, \cdots$, and $\tilde{\phi} = 0$. Thus $u'(x) = u^*(x)$ and

$\psi'(x,\mu) = \psi^*(x,\mu)$. Thus we complete the proof.

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