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OPTIMAL CONTROL APPLICATIONS IN TRANSPORT THEORY

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ABSTRACT

The critical neutron flux flattening problem governed by the transport equation is studied in a nonuniform slab with vacuum boundary conditions. Existence and uniqueness theorem of the optimal solution is shown in continuous function space. And it is shown that the continuous-space discrete-ordinates approximation converges to the exact optimal solution while the corresponding controllable function converges in the sense of weak star topology.

1. INTRODUCTION

Optimal control theory of distributed parameter systems has come into wide use in the nuclear field, both to nuclear design problems and to nuclear plant operation problems[1]. One type of the problems is how to design a reactor with the least amount of fuel. Many theoretical studies[2-5] concerning this subject have shown that the condition for minimum critical mass is a flattening of the neutron flux distribution in the fuel region. And a few experimental works[6] have been published to suggest that the condition is generally satisfied. The condition implies that it is a very interesting subject to explore ways to flatten the neutron flux in order to achieve, with a minimum amount of fuel, a critical assembly. In this paper, we are concerned with the neutron flux flattening problem governed by the steady-state transport equation in a nonuniform slab reactor with vacuum boundary conditions. We try to show that the neutron flux can be flattened by manipulating the fissile material distribution. First, we establish the mathematical model from the view of optimal control theory[7]. Then we show the existence-uniqueness theorem of the optimal solution. Lastly we consider the discrete-ordinates approximation of the optimal solution and show that it converges to the exact optimal solution in max norm while the corresponding controllable function converges in the sense of weak star topology.

2. FORMULATION OF THE PROBLEM

Consider the critical problem for mono-energetic neutron transport in a nonuniform slab of thickness 2a surrounded either by vacuum or by a perfect absorber. The neutron transport can be described by the following integral equation[8]

$$\phi = \int_{-a}^{a} K(x,x')u(x')\phi(x')dx', \tag{2.1}$$

$$K(x,x') = \frac{1}{2}\sigma(x') \int_{1}^{+\infty} \frac{dt}{t} \exp(-|\int_{x}^{x'} \sigma(s)ds|t), \tag{2.2}$$

where ϕ is the neutron flux, $\sigma(x)$ is the total cross section, u(x) is the mean number of secondaries per collision.

Let's choose the deviation of the flux distribution from its average value as the performance index[cf.1]. The smaller it is, the flatter the flux is. So the neutron flux flattening problem is to

$$\text{minimize } F(\phi) = \int_{-a}^{a} |\phi(x) - m\phi|^2 dx, \tag{2.3}$$

while the state variable $\phi \geq 0$ satisfies the integral equation (2.1) and makes all the fuel operate at the same specific power, which can be expressed by

$$\int_{-a}^{a} \sigma(x)u(x)\phi(x)dx = \frac{P}{2}, \tag{2.4}$$

where $m\phi$ denotes the mean value of ϕ, that is $m\phi = \frac{1}{2a}\int_{-a}^{a}\phi(x)dx$.

Remarks: What we mean by a function $g(\cdot) \geq 0$ is that $g(\cdot) \geq 0$ and there exists a positive measure set in which $g(\cdot) > 0$.

Let's take u(x) as the controllable function, and it belongs to the admissible control set denoted by

$$U \equiv \{u \in X : 0 \leq u(x) \leq C_0\},$$

where C_0 is a positive constant, and X $\equiv L^{\infty}[-a, a]$, the usual Banach space consisting of all essentially bounded functions on [-a, a].

Let Q = [-a, a], Z be the state space consisting of continuous functions in Q with the maximum norm $\|\cdot\| = \max_{x\in Q} |\cdot(x)|$, and W denote the set $\{\phi \in Z :$ there exists $u \in U$ such that (u, ϕ) satisfies (2.1) and (2.4), and $\phi \geq 0\}$. Then the neutron flux flattening problem can be written as the minimization problem

$$F(\phi^*) = \min_{\phi\in W} F(\phi), \qquad \phi^* \in W \tag{2.5}$$

3. EXISTENCE-UNIQUENESS THEOREM

In this paper, we assume that σ(x) is a positive continuous function on

[-a, a], and $0 < \sigma_m = \min_{x\in[-a,a]} \sigma(x) \leq \sigma(x) \leq \max_{x\in[-a,a]} \sigma(x) = \sigma_M$.

Considering the transform

$$y = f(x) = \int_{-a}^{x} \sigma(s)ds - \frac{\Delta_0}{2},\ \Delta_0 = \int_{-a}^{a} \sigma(s)ds \tag{3.1}$$

we can convert Eq.(2.1) into the following equivalent equation

$$\tilde{\phi}(y) = \frac{1}{2}\int_{-\frac{\Delta_0}{2}}^{\frac{\Delta_0}{2}} \int_{1}^{+\infty} \frac{dt}{t} \exp(-|y-y'|t)\tilde{u}(y')\tilde{\phi}(y')dy' \tag{3.2}$$

where $\tilde{\phi}(y) = \phi(f^{-1}(y))$ and $\tilde{u}(y) = u(f^{-1}(y))$. Hence, without loss of generality, we assume that $\sigma(x) = 1$ in the following.

Let us define integral operators K_1 from Z to Z, K_2 from X to Z, and K_3 from $L^2(Q)$ to $L^2(Q)$, the usual Hilbert space composed of square integrable functions, by

$$K_i\bullet = \int_{-a}^{a} K(x,x')\bullet(x')dx', \quad i = 1,2,3 \tag{3.3}$$

Lemma 3.1 (1) if $\phi = K_i\phi$ and $\phi \geq 0$, then $\phi > 0, i = 1,2,3$;

(2) K_1 is a compact operator from Z to Z;

(3) K_2 is a compact operator from X to Z.

(4) K_3 is a compact operator from $L^2(Q)$ to $L^2(Q)$.

Proof. (1) is obvious.

(2) and (3) may be shown by use of Arzela-Ascoli's Theorem(see, e.g.[9, p.248]).

(4) has been shown in Ref.10.

Lemma 3.2[cf. 11] W $\neq \emptyset$ iff $C_0 \geq \frac{1}{\gamma(K_1)}$, where $\gamma(K_1)$ denotes the spectral radius of K_1.

Proof. The same procedure as in Lemma 6 in Ref. 12 shows that $\gamma(K_1) > 0$. Theorem 2.1 in Ref. 13 shows that $\gamma(K_1)$ is an eigenvalue of K_1 with nonnegative eigenfunction ϕ. Let $\hat{\phi}(x) = \frac{P\gamma(K_1)\phi(x)}{2\int_{-a}^{a}\phi(x)dx}$, then $\hat{\phi} \in W$ with the corresponding $\hat{u}(x) \equiv \frac{1}{\gamma(K_1)}$.

On the other hand, if W $\neq \emptyset$, there exists u $\in$ U and $\phi > 0$ [Lemma 3.1(1)] such that $\phi = K_1 u\phi \leq MK_1\phi$. Therefore the same reason as in case of Lemma 6 in Ref. 12 shows that M$\geq \frac{1}{\gamma(K_1)}$.

Lemma 3.3 If $C_0 \geq \frac{1}{\gamma(K_1)}$, W is a bounded nonempty set in Z.

Proof. Since

$$K(x,x') = \frac{1}{2}\int_1^{+\infty} \frac{dt}{t}\exp(-|x'-x|t)$$
$$\leq \frac{1}{2}[\int_1^{+\infty}\exp(-2|x-x'|t)dt]^{1/2} = \frac{1}{2\sqrt{2}}\frac{1}{|x-x'|^{1/2}}, \tag{3.4}$$

there exists a constant C_1 such that

$$K^3(x,x') = \int_{-a}^{a}\int_{-a}^{a} K(x,x'')K(x'',x''')K(x''',x')dx''dx''' \leq C_1. \tag{3.5}$$

Hence for any $\phi \in W$,

$$\phi(x) = \int_{-a}^{a} K(x,x')u(x')\phi(x')dx'$$
$$= \int_{-a}^{a}[\int_{-a}^{a}\int_{-a}^{a} K(x,x'')u(x'')K(x'',x''')u(x''')K(x''',x')dx''dx''']u(x')\phi(x')dx'$$
$$\leq C_0^2 C_1 P/2. \tag{3.6}$$

This shows that W is bounded.

Lemma 3.4 If $C_0 \geq \frac{1}{\gamma(K_1)}$, W is a compact convex nonempty set in Z.

Proof. It is easy to show that W is a convex set in Z [cf. 11]. It follows from Lemma 3.1(3) and Lemma 3.3 that W is a locally compact set. So it is enough to show that W is closed.

Suppose that $\{\phi_n\} \subset$W, and ϕ_n strongly converges to $\phi_0 \in Z$. By the definition of W, there exists$\{u_n\} \subset$ U such that $\phi_n = K_3 u_n \phi_n$.

Since U is a weak* closed set in X = $(L^1(Q))^*$, the conjugate space of the usual absolutely integrable function space $L^1(Q)$, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u_0 \in U$ such that

$$\int_{-a}^{a} u_n(x)f(x)dx \to \int_{-a}^{a} u_0(x)f(x)dx, \text{ for every f} \in L^1(Q), \tag{3.7}$$

And it is easy to see that

$$\int_{-a}^{a} u_n(x)\phi_n(x)f(x)dx \to \int_{-a}^{a} u_0(x)\phi_0(x)f(x)dx, \text{ for every f} \in L^1(Q), \tag{3.8}$$

In particular,

$$\int_{-a}^{a} u_n(x)\phi_n(x)f(x)dx \to \int_{-a}^{a} u_0(x)\phi_0(x)f(x)dx, \text{ for every f} \in L^2(Q), \tag{3.9}$$

Lemma 3.1(4) concludes $\phi_0 = K_3 u_0 \phi_0$. That is

$$\phi_0(x) = \int_{-a}^{a} K(x,x')u_0(x')\phi_0(x')dx', \tag{3.10}$$

Therefore it is easy to find $\phi_0 \in W$.

Lemma 3.5 $F(\cdot)$ is a strictly convex continuous functional in W.

Proof. It is easy to show the continuity of F. Now let us show the strictly convexity of F.

For any ϕ_1, ϕ_2 in W, and $0 < t < 1$,

$$F(t\phi_1 + (1-t)\phi_2)$$
$$= \int_{-a}^{a}[t(\phi_1(x) - m\phi_1) + (1-t)(\phi_2(x) - m\phi_2)]^2 dx$$
$$= t\int_{-a}^{a}(\phi_1(x) - m\phi_1)^2 dx + (1-t)\int_{-a}^{a}(\phi_2(x) - m\phi_2)^2 dx$$
$$-t(1-t)\int_{-a}^{a}[(\phi_1(x) - m\phi_1 - (\phi_2(x) - m\phi_2)]^2 dx, \tag{3.11}$$

If $\int_{-a}^{a}[(\phi_1(x) - m\phi_1 - (\phi_2(x) - m\phi_2)]^2 dx = 0$, $(\phi_1(x) - m\phi_1) - (\phi_2(x) - m\phi_2) = 0$. So $\phi_1(x) - \phi_2(x) = m(\phi_1 - \phi_2) = constant$, and the boundary conditions conclude that $\phi_1(x) = \phi_2(x)$. Therefore we have shown the lemma.

From Lemma 3.4 and Lemma 3.5, one may easily obtain the following existence-uniqueness theorem.

Theorem 3.1 If $C_0 \geq \frac{1}{\gamma(K_1)}$, there is unique ϕ^* in W such that

$$F(\phi^*) = \min_{\phi \in W} F(\phi), \tag{3.12}$$

Corollary 3.1 If $C_0 \geq \frac{1}{\gamma(K_1)}$, there is a unique u^* in U and ϕ^* in W such that

$$\phi^* = \int_{-a}^{a} K(x, x')u^*(x')\phi^*(x')dx', \tag{3.13}$$

and $\phi^* > 0$, $\quad$ (3.14)

and $\int_{-a}^{a} u^*(x)\phi^*(x)dx = \frac{P}{2}$ $\quad$ (3.15)

and $F(\phi^*) = \min_{\phi \in W} F(\phi)$. $\quad$ (3.16)

Proof. It follows from [Theorem 2, Ref.11] that K_i^{-1}, i = 1, 2, 3, exist. The uniqueness and strict positiveness of ϕ^* in Theorem 3.1 concludes this corollary.

4. CONVERGENCE OF THE DISCRETE ORDINATES APPROXIMATION

In this section, we consider the discrete ordinates approximation to the original problem (2.5).

Regarding the quadrature rule we choose such quadrature points μ_{nj} and associated weights ω_{nj} that

$$\lim_{n\to\infty} \sum_{|j|=1}^{n} \omega_{nj} f(\mu_{nj}) = \int_{-1}^{1} f(\mu)d\mu \tag{4.1}$$

for any f continuous on [-1,1], and

$\omega_{n,-j} = \omega_{n,j} > 0, \mu_{n,-j} = -\mu_{n,j}, 0 < \mu_{n1} < \mu_{n2} < \cdots < \mu_{nn} = 1.$

Then one may easily check that the discrete problem analogous to (2.5) is to minimize

$$F(\phi_n) = \int_{-a}^{a} |\phi_n(x) - m\phi_n|^2 dx, \tag{4.2}$$

where $\phi_n = \frac{1}{2} \sum_{|j|=1}^{n} \omega_{nj} \psi_n(x, \mu_{nj})$ satisfies

$$\phi_n = \frac{1}{2} \int_{-a}^{a} \sum_{j=1}^{n} \omega_{nj} \frac{1}{\mu_{nj}} \exp(-\frac{1}{\mu_{nj}}|x' - x|) u(x') \phi_n(x') dx' \tag{4.3}$$

and $\phi_n(x) \geq 0$ (4.4)

and $\int_{-a}^{a} u(x)\phi_n(x)dx = \frac{P}{2}$. (4.5)

Let $W_n = \{\phi_n \in Z$: there exists $u_n \in U$ such that (u_n, ϕ_n) satisfies (4.3)-(4.5)$\}$. Then the discrete neutron flux flattening problem analogous to (2.5) can be written as the minimization problem

$$F(\phi_n^o) = \min_{\phi_n \in W_n} F(\phi_n), \qquad \phi_n^o \in W_n, \tag{4.6}$$

Let K_i^n analogous to K_i, i =1, 2, 3 be defined by

$$K_i^n \phi_n = \frac{1}{2} \int_{-a}^{a} \sum_{j=1}^{n} \omega_{nj} \frac{1}{\mu_{nj}} \exp(-\frac{1}{\mu_{nj}}|x' - x|) \phi_n(x') dx', \text{ i} = 1, 2, 3 \tag{4.7}$$

The existence-uniqueness theorem analogous to Corollary 3.1 can be stated by

Theorem 4.1 If $C_0 \geq \frac{1}{\gamma(K_1^n)}$, where $\gamma(K_1^n)$ denotes the spectral radius of K_1^n, there is unique u_n^o in U and ϕ_n^o in W_n such that

$$\phi_n^o = \frac{1}{2} \int_{-a}^{a} \sum_{j=1}^{n} \omega_{nj} \frac{1}{\mu_{nj}} \exp(-\frac{1}{\mu_{nj}}|x' - x|) u_n^o(x') \phi_n^o(x') dx', \tag{4.8}$$

and $\phi_n^o \geq 0$, (4.9)

and $\int_{-a}^{a} u_n^o(x)\phi_n^o(x)dx = \frac{P}{2}$ (4.10)

and $F(\phi_n^o) = \min_{\phi_n \in W_n} F(\phi_n)$. (4.11)

Now we wonder whether ϕ_n^o and u_n^o converge to ϕ^* and u^* as $n \to \infty$, respectively.

Lemma 4.1[14] K_i^n is collectively compact, and K_i^n converges uniformly to K_i as $n \to \infty$, i = 1, 2.

Lemma 4.2[15] For sufficiently large n, there exist real number λ_n and $\phi_n^* \in Z$ such that $\phi_n^* \geq 0$ and that

$$\lambda_n \phi_n^* = K_1^n u^* \phi_n^* \tag{4.12}$$

and that $\lambda_n \to 1$, $\phi_n^* \to \phi^*$, as n$\to \infty$.

Lemma 4.3 If u^* is in the interior of U, there exists a sequence $\{\widehat{\phi}_n^*\}$, such that $\widehat{\phi}_n^* \in W_n$ for sufficiently large n and $\widehat{\phi}_n^*$ converges strongly to ϕ^* as $n \to \infty$, where ϕ^* is as in Theorem 3.1.

Proof. If u^* is in the interior of U, then for sufficiently large n, $\frac{u^*}{\lambda_n} \in U$ and the sequence chosen by

$$\widehat{\phi}_n^* = \frac{P\phi_n^*}{2\int_{-a}^{a} \frac{u^*(x)}{\lambda_n} \phi_n^*(x)dx}, \tag{4.13}$$

is the desired one, where λ_n and ϕ_n^* are as in Lemma 4.3.

Theorem 4.2 Let ϕ_n^o and u_n^o be as in Theorem 4.1 and ϕ^* and u^* be as in Corollary 3.1. If u^* is in the interior of U, then there exists a subsequence of $\{\phi_n^o\}$, still denoted by $\{\phi_n^o\}$, and a subsequence of $\{u_n^o\}$, still denoted by $\{u_n^o\}$, such that ϕ_n^o converges to ϕ^* in max norm and u_n^o converges to u^* in weak star topology.

Proof. Since $\phi_n^o = K_2^n u_n^o \phi_n^o$, and $\{u_n^o \phi_n^o\}$ is bounded in X [cf. Lemma 3.3], the collectively compact of K_2^n concludes that a subsequence of $\{\phi_n^o\}$, still denoted by $\{\phi_n^o\}$, converges to some $\phi^o \in Z$.

Since U is a weak* closed set in X, there exists a subsequence of $\{u_n^o\}$, still denoted by $\{u_n^o\}$, and $u^o \in U$ such that

$$\int_{-a}^{a} u_n^o(x) f(x)dx \to \int_{-a}^{a} u^o(x) f(x)dx, \text{ for every } f \in L^1(Q). \tag{4.14}$$

(3.10) shows that

$$K_2 u_n^o \phi_n^o \to K_2 u^o \phi^o, \text{ as } n \to \infty \tag{4.15}$$

So

$$K_2^n u_n^o \phi_n^o = (K_2^n - K_2) u_n^o \phi_n^o + K_2 u_n^o \phi_n^o \to K_2 u^o \phi^o, \text{ as } n \to \infty \tag{4.16}$$

This shows that $\phi^o = K_2 u^o \phi^o$. Thus $\phi^o \in W$.

Therefore

$$\int_{-a}^{a} |\phi^o(x) - m\phi^o|^2 dx = \lim_{n\to\infty} \int_{-a}^{a} |\phi_n^o(x) - m\phi_n^o|^2 dx$$

$$\leq \lim_{n\to\infty} \int_{-a}^{a} |\widehat{\phi}_n^*(x) - m\widehat{\phi}_n^*|^2 dx = \int_{-a}^{a} |\phi^*(x) - m\phi^*|^2 dx \tag{4.17}$$

The uniqueness of the optimal solution ϕ^* concludes that $\phi^* = \phi^o$. And Corollary 3.1 concludes that $u^o = u^*$. Therefore the theorem is proved.

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