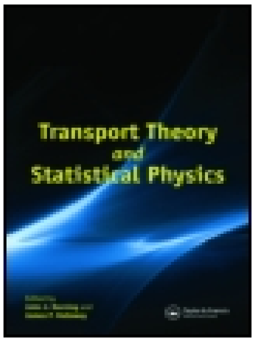

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COMPLEX EIGENVALUES OF A MONO-ENERGETIC NEUTRON TRANSPORT OPERATOR

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ABSTRACT

The complex eigenvalue problem of a mono-energetic neutron transport operator is studied in a homogeneous sphere with spherically symmetric scattering. It is shown that the spectrum involves a countable infinity of complex eigenvalues.

1. Introduction

Let S be a sphere of radius a surrounded by the vacuum. Then the mono-energetic neutron transport operator with spherically symmetric scattering is defined as

$$A\psi = -\mu\frac{\partial}{\partial r}\psi - \frac{1-\mu^2}{r}\frac{\partial\psi}{\partial\mu} - \Sigma(r)\psi(r,\mu) + \frac{c(r)}{2}\int_{-1}^{1}\psi(r,\mu')d\mu' \tag{1}$$

$$\psi \in D(A) = \{\psi \in H : 1)A\psi \in H;\ 2)\psi(a,\mu) = 0\ if\ \mu < 0\} \tag{2}$$

where $\Sigma(r)$ denotes the total cross section, $c(r)$ denotes the scattering cross section, $\psi(r,\mu)$ is the angular density depending on r, the distance from the center of the sphere and μ is the cosine of the angle the neutron beam makes with the position vector $\mathbf{r}$. H is the Hilbert space of square summable functions with its inner product

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$$(\psi,\varphi) = 4\pi \int_0^a r^2 dr \int_{-1}^1 \psi(r,\mu)\overline{\varphi(r,\mu)}d\mu \tag{3}$$

First, let us review some results of the spectral analysis of A in the following two cases:

c1) $\Sigma(r) = \Sigma$ (positive constant), and $c(r) = c$ (positive constant), for $r \le a$;

c2) $\Sigma(r) = \Sigma$ (positive constant), and

$$c(r) = \begin{cases} c, & if\ a-d \le r \le a, \\ 0, & if\ 0 \le r < a-d \end{cases}$$

It is well known[1] that in all cases the spectrum $\sigma(A)$ of the operator A consists of discrete eigenvalues with $-\infty$ as the only possible accumulation point.

In 1962 R. Van Norton[2] further considered the existence of the real eigenvalues in the case c1). It is shown that the operator has an infinity of real eigenvalues accumulating at $-\infty$. In 1964 Tian Fangzeng[3] showed that the eigenvalues belonging to the half-plane $\mathrm{Re}\lambda \ge -\Sigma$ must be real. However, whether complex eigenvalues in the other half-plane exist is not considered up to now, because of the well-known mathematical difficulty.

In 1976 B. Montagnini[4] considered the existence of the complex eigenvalues in the case c2 and proved the following theorem.

Theorem 1.1 Let Λ_k be the rectangle in the complex plane with sides parallel to the axes and the points

$$-\Sigma - \frac{1}{a}\ln\frac{k\pi}{\sqrt{2ca}} + i\frac{k\pi}{a}, \quad -\Sigma - \frac{1}{a}\ln\frac{12k\pi}{\sqrt{2ca}} + i\frac{(k+1)\pi}{a},$$

as opposite vertices. Then there exists a real number d_0 , with $0 < d_0 < a$, and an integer $k_0 > 0$ such that for $d < d_0$, $k > k_0$, each Λ_k contains one eigenvalue of A.

In 1993 Xu Bang Qing[5] improved the relations in Ref. 4, and removed the restriction $d < d_0$. His results can be stated as

Theorem 1.2 Let Λ_k be the rectangle in the complex plane with sides parallel to the axes and the points

$$-\Sigma - \frac{1}{a}\ln\frac{k\pi}{\sqrt{2ca}} + i\frac{k\pi}{a}, \quad -\Sigma - \frac{1}{a}\ln\frac{12k\pi}{\sqrt{2ca}} + i\frac{(k+1)\pi}{a},$$

as opposite vertices. Then for any $0 < d < a$ there exists integer $k_d > 0$ such that for $k > k_d$, each Λ_k contains one eigenvalue of A.

The procedure in Ref. 4 begins with the following so-called asymptotic integral equation

$$\phi(x) = 2\zeta r(\zeta)\int_0^\delta G(x,y;\zeta)\phi(y)dy \tag{4}$$

$$r(\zeta) = \frac{ca}{2}\frac{e^\zeta}{\zeta^2} \tag{5}$$

$$G(x,y;\zeta) = \frac{e^{-\zeta(x+y)}}{1-x-y} \tag{6}$$

where $\phi(x)$ belongs to $L^2(0,\delta)$, $\delta = \frac{d}{2a}$. Only if $\delta < \frac{1}{2}$ (i.e., $d < a$), can the asymptotic integral equation (4) be transformed into a linear system with infinitely many unknowns. Therefore, the procedure cannot be applied to the case c1). In this paper, we use Theorem 1.2 as a bridge to show that the operator A in case c1) involves a countable infinity of complex eigenvalues.

2. Preliminary

In this paper, we suppose $\Sigma(r) = \Sigma$. Then we can further assume that $\Sigma = 0$, without loss of generality, since the spectra of A and $A - \Sigma I$ can be obtained from each other by translation, where I is the identity operator.

Just as in Ref. 2, upon the transformation of variables $x = r\mu, y = r\sqrt{1-\mu^2}$, the operator A can be expressed by

$$Af = -\frac{\partial f}{\partial x} - \frac{c(r)}{2r}\int_{-r}^{r} f(z,\sqrt{r^2-z^2})dz \tag{7}$$

$$f \in D(A) = \{f \in H' : 1)\frac{\partial f}{\partial x} \in H'; 2) f(-\sqrt{a^2-y^2}, y) = 0\} \tag{8}$$

where H' is the Hilbert space consisting of square integrable functions defined for $r^2 = x^2 + y^2 \le a^2$ and $y \ge 0$, and its inner product defined to be

$$(f,g) = 4\pi\int_0^a ydy\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x,y)\overline{g(x,y)}dx \tag{9}$$

Suppose $0 < d_n < a$, $n = 1,2,\ldots, d_n \to a$, as $n \to \infty$. We denote the operator A by A_n, while

$$c(r) = \begin{cases} c, & if\ a - d_n \le r \le a, \\ 0, & if\ 0 \le r < a - d_n \end{cases}$$

and we denote the operator A by A_0, while $c(r) = c$, for $0 \le r \le a$.

Now let us consider the eigenvalue problem

$$A_n f = \lambda f, \quad n = 0, 1, 2, \ldots \tag{10}$$

Let

$$\phi(r) = \int_{-r}^{r} f(z, \sqrt{r^2 - z^2})dz \tag{11}$$

If f is in $D(A)$ and satisfies (10), then

$$\lambda f(x, y) + \frac{\partial f}{\partial x} = \frac{c(r)}{2r}\phi(r) \tag{12}$$

By integrating the equation , we obtain

$$f(x, y) = \frac{1}{2}e^{-\lambda x}\int_{-\sqrt{a^2-y^2}}^{x} c(\sqrt{z^2+y^2})\phi(\sqrt{z^2+y^2})dz \tag{13}$$

and

$$\phi(r) = \frac{1}{2}\int_0^a c(s)\phi(s)ds\int_{|r-s|}^{r+s} e^{-\lambda\omega}\frac{d\omega}{\omega} \tag{14}$$

Let $h = L^2(0, a)$ be the usual Hilbert space, $L_{\lambda,n}$ be the integral operator defined by

$$L_{\lambda,n}\phi = \int_0^a L_n(r, s; \lambda)\phi(s)ds, \quad n = 0, 1, 2, \ldots \tag{15}$$

$$L_0(r, s; \lambda) = \int_{|r-s|}^{r+s} e^{-\lambda\omega}\frac{d\omega}{\omega}$$

$$L_n(r, s; \lambda) = \begin{cases} L_0(r, s; \lambda), & if \;\; a - d_n \le r \le a, \\ 0, & if \;\; 0 \le r < a - d_n \end{cases} \quad n = 1, 2, \ldots$$

Since $\phi(r)$ defined by (11) is in h, (14) can be expressed by the operator equation

$$\phi = \frac{c}{2}L_{\lambda,n}\phi, \quad n = 0, 1, 2, \ldots \tag{16}$$

Literally applying Theorem 2.1 and Theorem 2.2 in Ref.2 to our cases, we obtain

Theorem 2.1 (1) $L_{\lambda,n}$, $n = 0, 1, 2, \ldots$, are compact operators in h;
(2) The necessary and sufficient condition for $f_n(x, y)$ to be an eigenfunction of A_n, $n = 0, 1, 2, \ldots$, is that its associated ϕ_n, $n = 0, 1, 2, \ldots$, is an eigenfunction of $L_{\lambda,n}$ with the eigenvalue $2/c$.

Before ending this section, let us refer to [6] for an important conclusion as our Theorem 2.2.

Theorem 2.2 The operator A_n, $n = 0, 1, 2, ...$, has the dominant eigenvalue $\beta_0^{(n)}$ such that $\beta_0^{(n)} > \beta_0^{(0)}$, $n = 1, 2, ...$

3. Complex Eigenvalues of A_0

First let us refer to [7] for some notations and a useful theorem. Suppose that X is a Banach space and $B = \{x \in X : \|x\| \leq 1\}$ is the closed unit ball in X. Let $[X]$ be the Banach space of all bounded linear operators on X.

Definition 3.1: A subset G in $[X]$ is said to be collectively compact if the closure of the set $\{Tx : T \in G, x \in B\}$ is compact in X.

Definition 3.2). A sequence of operators $\{T_n\}$ in $[X]$ is p-order quasi-collectively compact if it has the following properties:

(i) $T_n^p = K_n + R_n$, $n = 1, 2, 3, ..$, and p is some positive integer;

(ii) $\{K_n\}$ is collectively compact;

(iii) $\lim_{n\to\infty} \|R_n\| = 0$.

For simplicity, $\{T_n\}$ is called quasi-collectively compact if $p = 1$.

Theorem 3.1 Let T, T_n be in $[X]$, $\{T_n\}$ quasi-collectively compact, and T_n strongly convergent to T. If λ_n is the eigenvalue of T_n, and $\lambda_n \to \lambda_0 \neq 0$, then λ_0 is the eigenvalue of T.

In the following, we denote $E_\tau = \{\lambda \in C : Re\lambda \geq \tau\}, \tau < 0$

Lemma 3.1 (i) $L_{\lambda,n}$ converges to 0 in operator norm uniformly in n, as $Re\lambda \to +\infty$;

(ii) $L_{\lambda,n}$ converges to 0 in operator norm uniformly in n, as $|Im\lambda| \to +\infty$, when $\lambda \in E_\tau$.

Proof: It is easy to show by Schwartz inequality

$$\|L_{\lambda,n}\|^2 \leq \int_0^a \int_0^a |L_0(r, s; \lambda)|^2 dr ds \tag{17}$$

when λ in E_τ,

$$|L_0(r, s; \lambda)| \leq \int_{|r-s|}^{r+s} \frac{e^{-\tau\omega}}{\omega} d\omega \leq \int_{|r-s|}^{2a} \frac{e^{-2\tau a}}{\omega} d\omega = e^{-2\tau a} \ln \frac{2a}{|r-s|}$$

Since $\ln \frac{1}{u} = 0(\frac{1}{u^\alpha})$, as $u \to 0$, $0 < \alpha < 1/2$,

$$\int_0^a \int_0^a |L_0(r, s; \lambda)|^2 dr ds \leq \int_0^a \int_0^a (\ln \frac{2a}{|r-s|})^2 dr ds < +\infty \tag{18}$$

(i) Since

$$|L_0(r,s;\lambda)| \le \int_{|r-s|}^{r+s} \frac{e^{-Re\lambda\omega}}{\omega} d\omega = \int_{|r-s|}^{r+s} e^{-Re\lambda\omega} \int_0^\infty e^{-\omega v} dv$$

$$= \int_{Re\lambda}^\infty \frac{dv}{v}(e^{-v|r-s|} - e^{-v(r+s)}) \tag{19}$$

and

$$\int_0^\infty \frac{dv}{v}(e^{-v|r-s|} - e^{-v(r+s)}) = \ln \frac{r+s}{|r-s|},$$

so

$$\lim_{Re\lambda \to +\infty} |L_0(r,s;\lambda)| = 0 \tag{20}$$

From (18), (20) and by Lebesgue Theorem, $\|L_{\lambda,n}\| \to 0$, uniformly in n, as $Re\lambda \to +\infty$

(ii) We have

$$L_0(r,s;\lambda) = \int_{|r-s|}^{r+s} \frac{e^{-\lambda\omega}}{\omega} d\omega = \int_0^\infty \frac{dv}{\lambda+v}(e^{-(\lambda+v)|r-s|} - e^{-(\lambda+v)(r+s)})$$

By use of the formula:

$$\int_0^\infty \frac{e^{-\omega}}{z-\omega} d\omega = \frac{1}{z}(1 + \frac{1}{z} + \frac{1}{z}\int_0^\infty \frac{\omega^2 e^{-\omega}}{z-\omega} d\omega),\ (Imz \ne 0),$$

we obtain

$$\int_0^\infty \frac{dv}{\lambda+v} e^{-(\lambda+v)|r-s|} = \frac{e^{-\lambda|r-s|}}{\lambda|r-s|}(1 - \frac{1}{\lambda|r-s|} + \frac{1}{\lambda|r-s|}\int_0^\infty \frac{\omega^2 e^{-\omega}}{\lambda|r-s|+\omega} d\omega)$$

It is easy to calculate that

$$|\int_0^\infty \frac{\omega^2 e^{-\omega}}{\lambda|r-s|+\omega} d\omega| \le \frac{2}{|Im\lambda||r-s|}$$

and when λ in E_τ,

$$|e^{-\lambda|r-s|}| \le e^{-\tau|r-s|}$$

So

$$|\int_0^\infty \frac{dv}{\lambda+v} e^{-(\lambda+v)|r-s|}| \to 0,\ as\ |Im\lambda| \to +\infty$$

By the same process, we know that

$$|\int_0^\infty \frac{dv}{\lambda+v} e^{-(\lambda+v)(r+s)}| \to 0,\ as\ |Im\lambda| \to +\infty$$

Therefore,

$$|L_0(r,s;\lambda)| \to 0, \ as \ |Im\lambda| \to +\infty \tag{21}$$

From (18), (21) and by Lebesgue Theorem, $\|L_{\lambda,n}\| \to 0$, uniformly in n, as $|Im\lambda| \to +\infty$

Theorem 3.2 Let $\tau(A_n) = \sigma(A_n) \cap E_\tau, \tau < \min(0, \beta_0^{(0)})$. Then $\tau(A_n) \neq \emptyset$, and $S = \{\lambda \in C | \lambda \in \tau(A_n), n = 1,2,3,...\}$ is a bounded set.

Proof: $\tau(A_n) \neq \emptyset$, as $\tau < \min(0, \beta_0^{(0)})$, is a direct consequence of Theorem 2.2. For any sequence $\{\mu_{n_k}\} \subset S$, $\mu_{n_k} \in \sigma(A_{n_k})$, by Theorem 2.1, there exists ϕ_{n_k} in h, such that

$$L_{\mu_{n_k},n_k}\phi_{n_k} = \frac{2}{c}\phi_{n_k}$$

So

$$\|L_{\mu_{n_k},n_k}\| \geq \frac{2}{c}$$

By Lemma 3.1, $Re\mu_{n_k} < +\infty, |Im\mu_{n_k}| < +\infty$. So the claim is true.

Theorem 3.3 Let $\{\mu_{n_k}\} \subset S, \mu_{n_k} \to \mu_0$, as $k \to \infty$. Then

(i) $\{L_{\mu_{n_k},n_k}\}$ is quasi-collectively compact;

(ii) $\{L_{\mu_{n_k},n_k}\}$ strongly converges to $L_{\mu_0,0}$;

(iii) $\mu_0 \in \sigma(A_0)$.

Proof: Let

$$K_k\phi = \int_0^a L_{n_k}(r,s;\mu_0)\phi(s)ds$$

$$R_k\phi = \int_0^a (L_{n_k}(r,s;\mu_{n_k}) - L_{n_k}(r,s;\mu_0))\phi(s)ds$$

Then

$$L_{\mu_{n_k},n_k}\phi = \int_0^a L_{n_k}(r,s;\mu_{n_k})\phi(s)ds = K_k\phi + R_k\phi$$

Since

$$\lim_{k\to\infty} \frac{e^{-\mu_{n_k}\omega} - e^{-\mu_0\omega}}{\omega} = 0,$$

and

$$\left|\frac{e^{-\mu_{n_k}\omega} - e^{-\mu_0\omega}}{\omega}\right| \leq \frac{e^{-Re\mu_{n_k}\omega}}{\omega} + \frac{e^{-Re\mu_0\omega}}{\omega}$$

so from (19) and by Lebesgue Theorem,

$$L_0(r,s;\mu_{n_k}) - L_0(r,s;\mu_0) = \int_{|r-s|}^{r+s} \frac{e^{-\mu_{n_k}\omega} - e^{-\mu_0\omega}}{\omega}d\omega \to 0,$$

and combining with (18),

$$\|R_k\|^2 \le \int_0^a \int_0^a |L_0(r,s;\mu_{n_k}) - L_0(r,s;\mu_0)|^2 dr ds \to 0 \tag{22}$$

For the same reason with [8], we know that $\{K_k\}$ is collectively compact, so (i) is true.

(ii) for any ϕ in X, let us denote

$$I_k\phi = \int_0^a (L_{n_k}(r,s;\mu_{n_k}) - L_0(r,s;\mu_{n_k}))\phi(s)ds$$

$$I'_k\phi = \int_0^a (L_0(r,s;\mu_{n_k}) - L_0(r,s;\mu_0))\phi(s)ds$$

Then

$$\|I_k\phi\|^2 \le \int_0^a \int_0^a |L_0(r,s;\lambda)|^2 dr ds \int_0^{a-d_{n_k}} |\phi(s)|^2 ds \to 0,$$

and by (22), $\|I'_k\phi\| \to 0,\ as\ k \to \infty$

(iii) By Theorem 2.1, $2/c$ is an eigenvalue of $L_{\mu_{n_k},n_k}$ By Theorem 3.1, $2/c$ is an eigenvalue of $L_{\mu_0,0}$, also. So μ_0 is an eigenvalue of A_0.

Theorem 3.4 For any $\varepsilon > 0$, there exists a positive integer N such that if $n > N$,

$$\sup_{y\in\tau(A_n)} \inf_{x\in\tau(A_0)} |y-x| < \varepsilon,\ \ for\ \ any\ \ \tau < \min(0,\beta_0^{(0)})$$

Proof: If the conclusion is false, there exists some $\varepsilon_0 > 0$, $\tau_0 < \min(0,\beta_0^{(0)})$, and an integer sequence $\{n_k\}$ such that

$$\sup_{y\in\tau_0(A_{n_k})} \inf_{x\in\tau_0(A_0)} |y-x| \ge \varepsilon_0,\ k = 1,2,...$$

And there exists $\{y_{n_k}\} \in \tau_0(A_{n_k})$, such that

$$\inf_{x\in\tau_0(A_0)} |y_{n_k} - x| > \varepsilon_0 - \frac{1}{k},\ k = 1,2,... \tag{23}$$

By Theorem 3.2, there exists a convergent subsequence of $\{y_{n_k}\}$, still denoted by $\{y_{n_k}\}$, with $y_{n_k} \to y_0$. Then

$$\inf_{x\in\tau_0(A_0)} |y_0 - x| \ge \varepsilon_0$$

so $|y_0 - x| \ge \varepsilon_0$, for any $x \in \tau_0(A_0)$. This contradicts (iii) in Theorem 3.3.

Corollary The operator A_0 has a countable infinity of complex eigenvalues.

Proof: Letting $\varepsilon < \frac{\pi}{3a}$, there exists M such that if $n > M$,

$$\sup_{y\in\tau(A_n)} \inf_{x\in\tau(A_0)} |y - x| < \varepsilon, \quad for \quad any \quad \tau < \min(0, \beta_0^{(0)})$$

Since $\tau(A_0)$ is a finite set, for any $\lambda \in \tau(A_n)$, there exist $x_\lambda \in \tau(A_0)$ such that $|\lambda - x_\lambda| < \varepsilon$.

Let $\lambda_{k+k_0} \in \Lambda_{k+k_0}$, $k = 1, 2, \ldots$ be the infinite eigenvalues of A_{M+1} chosen as in Theorem 1.2. Then $x_{\lambda_{2k+k_0}}$, $k = 1, 2, \ldots$, are the infinite eigenvalues of A_0. So our claim is proved.

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